Answer on Question #81208 – Math – Differential Equations
Question

Solve

$$p^2 + 2py\cot x = y^2$$

Solution

$p^2 + 2py\cot x = y^2$
$\cot^2 x + 1 = \operatorname{cosec}^2 x => \operatorname{cosec}^2 x - \cot^2 x = 1$
$p^2 + 2py\cot x = y^2(\operatorname{cosec}^2 x - \cot^2 x)$
$(p^2 + 2py\cot x + y^2\cot^2 x) - y^2\operatorname{cosec}^2 x = 0$
$(p + y\cot x)^2 - y^2\operatorname{cosec}^2 x = 0$
$(p + y\cot x + y\operatorname{cosec} x)(p + y\cot x - y\operatorname{cosec} x) = 0$
$p + y\cot x + y\operatorname{cosec} x = 0 \quad \text{or} \quad p + y\cot x - y\operatorname{cosec} x = 0$

$p + y\cot x + y\operatorname{cosec} x = 0$
$\frac{dy}{dx} + y(\cot x + \operatorname{cosec} x) = 0$
$\frac{dy}{y} = -\left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right)dx$
$\frac{dy}{y} = -\left(\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right)dx$
$\int\frac{dy}{y} = -\int\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}dx$

$\int\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}dx$

Substitution

$u = \sin\left(\frac{x}{2}\right), du = \frac{1}{2}\cos\left(\frac{x}{2}\right)dx$

$\int\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}dx = 2\int\frac{1}{u}du = 2\ln|u| - \ln C_1 = 2\ln\left|\sin\left(\frac{x}{2}\right)\right| - \ln C_1$

$\ln y = -2\ln\left|\sin\left(\frac{x}{2}\right)\right| + \ln C_1$

$y = \frac{C_1}{\sin^2\left(\frac{x}{2}\right)} = C_1\operatorname{cosec}^2\left(\frac{x}{2}\right)$

$y\sin^2\left(\frac{x}{2}\right) = C_1$

$$p + y\cot x - y\operatorname{cosec} x = 0$$
$$\frac{dy}{dx} + y(\cot x - \operatorname{cosec} x) = 0$$
$$\frac{dy}{y} = \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) dx$$
$$\frac{dy}{y} = \left(\frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right) dx$$
$$\int \frac{dy}{y} = \int \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} dx$$

$$\int \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} dx$$

Substitution

$$u = \cos\left(\frac{x}{2}\right), du = -\frac{1}{2}\sin\left(\frac{x}{2}\right) dx$$
$$\int \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} dx = -2\int \frac{1}{u} du = -2\ln|u| + \ln C_2 = -2\ln\left|\cos\left(\frac{x}{2}\right)\right| + \ln C_2$$
$$\ln y = -2\ln\left|\cos\left(\frac{x}{2}\right)\right| + \ln C_2$$
$$y = \frac{C_2}{\cos^2\left(\frac{x}{2}\right)} = C_2\sec^2\left(\frac{x}{2}\right)$$
$$y\cos^2\left(\frac{x}{2}\right) = C_2$$

The general solution is

$$\left(y\sin^2\left(\frac{x}{2}\right) - C_1\right)\left(y\cos^2\left(\frac{x}{2}\right) - C_2\right) = 0.$$

Answer: $\left(y\sin^2\left(\frac{x}{2}\right) - C_1\right)\left(y\cos^2\left(\frac{x}{2}\right) - C_2\right) = 0.$