Answer on Question #81206 – Math – Discrete Mathematics

Question

Find the number of equivalence relations that can be defined on a set of 6 elements.

Solution

The number to find is the number of different partitions of a set of 6 elements into equivalence classes.

Let A_n denote the number of partitions for a set of n elements. We need to find A_6.

We will find a recursive formula for A_n. First, $A_0 = 1$ (no elements, one way), $A_1 = 1$ (one element, one way).

For the set of n element take one particular element. There are the following possibilities:

- this element is not equivalent to any other – there will be A_{n-1} such partitions (we take any partition of other n-1 elements and add a class with this one particular element to it)
- this element is in equivalence class with exactly one other element – there are $\binom{n-1}{1}$ ways to choose this element and A_{n-2} partitions of all other elements corresponding to each choice, totally $\binom{n-1}{1} A_{n-2}$ partitions
- this element is in equivalence class with exactly two other elements – there are $\binom{n-1}{2}$ways to choose these elements and A_{n-3} partitions of all other elements corresponding to each choice, totally $\binom{n-1}{2} A_{n-3}$ partitions

  ...
- this element is in equivalence class with all other elements – there are $\binom{n-1}{n-1}$ ways to choose these other elements and A_0 partitions of all other elements corresponding to each choice, totally $\binom{n-1}{n-1} A_0 = 1$ partitions

Thus, the formula is

$$A_n = A_{n-1} + \binom{n-1}{1} A_{n-2} + \binom{n-1}{2} A_{n-3} + \ldots + \binom{n-1}{n-2} A_1 + A_0$$

It follows from the formula that

$A_2 = A_1 + A_0 = 2$,

$A_3 = A_2 + 2A_1 + A_0 = 2 + 2 \cdot 1 + 1 = 5$,

$A_4 = A_3 + 3A_2 + 3A_1 + A_0 = 5 + 3 \cdot 2 + 3 \cdot 1 + 1 = 15,$

$A_5 = A_4 + 4A_3 + 6A_2 + 4A_1 + A_0 = 15 + 4 \cdot 5 + 6 \cdot 2 + 4 \cdot 1 + 1 = 52$

$$A_6 = A_5 + 5A_4 + 10A_3 + 10A_2 + 5A_1 + A_0 = 52 + 5 \cdot 15 + 10 \cdot 5 + 10 \cdot 2 + 5 \cdot 1 + 1 = 203.$$

Answer: 203 equivalence relations.