Answer on Question #81194 – Math – Statistics and Probability

Question

Use the Neyman-Pearson Lemma to obtain the best critical region for testing

H0: mu=0 against H1: mu<>0 in the case of a normal population N(mu, sigma^2), where sigma^2 is known. Hence find the power of the test.

Solution

Consider the relation of likelihood

$$\frac{L_1(\vec{x})}{L_0(\vec{x})} = \frac{\prod_{i=1}^{n} \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)}{\prod_{i=1}^{n} \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)} \exp\left(-\frac{x_i{}^2}{2\sigma^2}\right)} = \prod_{i=1}^{n} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2} + \frac{x_i{}^2}{2\sigma^2}\right) = \exp\left(\frac{2n\mu\overline{x} - n\mu^2}{2\sigma^2}\right)$$

Then the critical region by Neyman-Pearson lemma will be of the form

$$\frac{2n\mu\bar{x} - n\mu^2}{2\sigma^2} > C_0$$

If $\mu > 0$ it yields $\overline{x} > C_1$, if $\mu < 0$ it yields $\overline{x} < C_2$.

Totally the critical region has form $\overline{x} \in (-\infty; C_2) \cup (C_1; +\infty)$. We take $C_2 = -C_1$ because in this case the length of the interval when H_0 is accepted is maximum. So the critical area has a form $|\overline{x}| > C$. We find C from the equation

$$P_0(|\overline{x}| > C) = \alpha.$$

Under the hypothesis H_0 the distribution of $\overline{x}$ is $N\left(0, \frac{\sigma^2}{n}\right)$. Then

$$P_0(|\overline{x}| > C) = P_0\left(\left|\frac{\overline{x}\sqrt{n}}{\sigma}\right| > \frac{C\sqrt{n}}{\sigma}\right) = P_0\left(|z| > \frac{C\sqrt{n}}{\sigma}\right) = 2F\left(-\frac{C\sqrt{n}}{\sigma}\right).$$

We have an equation

$$2F\left(-\frac{C\sqrt{n}}{\sigma}\right) = \alpha ,$$

from which

$$C = -\frac{\sigma F^{-1}(\alpha/2)}{\sqrt{n}}$$

If $|\overline{x}| > C$ the hypothesis H_0 is declined, otherwise it is accepted.

The power of the test is

$$P_1(|\overline{x}| > C) = 1 - P_1(-C < \overline{x} < C) = 1 - P_1\left(\frac{-C-\mu}{\sigma\sqrt{n}} < \frac{\overline{x}-\mu}{\sigma\sqrt{n}} < \frac{C-\mu}{\sigma\sqrt{n}}\right) =$$

$$= 1 - \left(F\left(\frac{C-\mu}{\sigma\sqrt{n}}\right) - F\left(\frac{-C-\mu}{\sigma\sqrt{n}}\right)\right)$$

This value is a minimum for $\mu = 0$, hence the power of the test is $1 - \alpha$.