# ANSWER on Question #81170 – Math – Abstract Algebra

## QUESTION

Prove that if $G \neq \{e\}$ and $G$ has no proper non-trivial subgroup, then $G$ is finite and $o(G)$ is a prime number.

## SOLUTION

Suppose by contradiction: If $G$ has no nontrivial subgroups $\Rightarrow$ $G$ is infinite or $|G| \neq p$.

1 case: $G$ is infinite.

Let us prove that if $G$ is an infinite group then $G$ has infinitely many subgroups.

**Proof**. Let's consider the following set: $C = \{\langle g \rangle : g \in G\}$ - collection of all cyclic subgroups in $G$ generated by elements of $G$. Two cases are possible:

1) There exist infinitely many distinct cyclic subgroups $\Rightarrow$ We are done.

2) There exist finitely many distinct cyclic subgroups, for example $C = \{H_1, H_2, \dots, H_n\}$. Then

$$G = \bigcup_{i=1}^{n} H_i$$

Since $G$ is infinite then without loss of generality suppose that $H_1$ is also infinite, where $H_1 = \langle g_1 \rangle$. Let's consider the following set $\{\langle g_1^n \rangle : n \in N\}$ , the collection of all cyclic subgroups of $H_1 \subset G$.

Let $K_1 = \langle g_1 \rangle$, $K_2 = \langle g_1^2 \rangle$, $K_3 = \langle g_1^3 \rangle$, …….

It's easy to show that $K_n$ and $K_m$ are distinct for $n \neq m$. Indeed, without loss of generality take $n < m$ and taking $g_1^n \in K_n$ but $g_1^n \notin K_m$ otherwise $g_1^n = g_1^{ml}$, where $l \in Z \Rightarrow g_1^{n-ml} = e$ and since $H_1$ is infinite $\Rightarrow$ $n = ml$ which is a contradiction since $m > n$.

Thus, the subgroups $K_n$ for any $n \in N$ are cyclic subgroups of $H_1$ $\Rightarrow$ cyclic subgroups of $G$. **Q.E.D.**

2 case: $|G| \neq p$.

Suppose that $|G| = n$ where $n$ is composite $\Rightarrow n = pm$ where $p$ - prime and $m \geqslant 2$.

Let $a \in G$ such $a \neq e$ then $a^{pm} = e$.

1. If $a^p \neq e$ then considering the cyclic group of $G$, namely $H = \langle ap \rangle \Rightarrow 1 < |H| \leqslant m < pm$ this is a contradiciton.
2. If $a^p = e$ then we know that $a \neq e$ and considering the cyclic subgroup of $G$, namely $H = \langle a \rangle \Rightarrow 1 < |H| \leqslant p < pm$.

So we get two contradiction and it follows that $G$ is finite and its order is prime.

**Q.E.D.**