Answer on Question #81169 – Math – Abstract Algebra

Question

How many Sylow 5-subgroups, Sylow 3-subgroups and Sylow 2-subgroups can a group of order 200 have? Give reasons for your answers.

Solution

Case p=3: Firstly, we write $|G| = 200 = 2^3 5^2$. As order 3 does not divide order of G, there is no element of order 3. There is no 3-Sylow subgroup.

Case p= 5: Let P be a Sylow 5-subgoup of G. It exists by Sylow Theorem 1.

Let $N_G(P)$ be a normalizer of P and n_5 be the number of 5-Sylow subgroups in G.

By Sylow Orbit-Stabilizer Theorem, $n_5 = |G: N_G(P)|$.

By Sylow Theorem 3, $(n_5 - 1) \vdots 5$.

As $P \leq N_G(P)$, $|N_G(P)| \vdots |P|$.

So, 25 divides $N_G(P)$ and possible values of $|G: N_G(P)| = |G|:|N_G(P)| = 200: |N_G(P)|$ are 1, 2, 4, 8.

The only possibility is n_5 =1. By the way, that provide as with $N_G(P) = G$ and P is normal in G.

Theorem (Sylow III). For each prime p, let n_p be the number of p-Sylow subgroups of G, $|G|=p^k m=5^2 2^3$, where p=5 doesn't divide m=8. Then n_p=1 mod p=1 mod 5 and n_p | m, n_p | 8.

The only possibility for n_p=1 mod 5 and n_p | 8 is n_p=1. There is one 5-Sylow subgroup.

Case p=2:

Let Q be a Sylow 2-subgoup of G and n_2 number of 2-Sylow subgroups in G.

By Sylow Orbit-Stabalizer theorem, $n_2 = |G: N_G(Q)|$.

As $Q \leq N_G(Q)$, $|N_G(Q)| \vdots 8$. Thus, possible values for n_2 are 1, 5, 25.

Construction for $\boldsymbol{n_2 = 1}$*:* If G any Abelian group, i.e. cyclic group, then $\boldsymbol{n_2}$**=1.**

Construction for $\boldsymbol{n_2 = 25}$*:* Consider Dihedral group D_{25} , the group of symmetries of a regular 25-gon.

In a way similar as above, we can show that its Sylow 5-subgroup is normal. Hence, it is unique (corollary Sylow Theorem 2). Let's consider remaining elements. Their orders cannot have 5 as a divisor as all such elements are in a unique 5-Sylow subgroup. Thus, their order is 2. There are

$|D_{25}| - 25 = 25$ such elements. So, they are all conjugate to each other (also Sylow Theorem 2). So, there are 25 Sylow 2-subgroups in D_{25}.

Finally, consider $G = D_{25} \times \mathbb{Z}_4$. If P_2 is a $2 - \text{Sylow subgroup in } D_{25}$, $R \coloneqq P_2 \times \mathbb{Z}_4$ is $2 - \text{Sylow subgroup in G}$. Cardinal of orbit of R in *G* is the same as the cardinal of orbit of $P_2\ in D_{25}$. This is true as the direct product of D_{25} and $\mathbb{Z}_4$. Thus, $\boldsymbol{n_2 = 25}$ for G.

Construction for $\boldsymbol{n_2 = 5}$*:* $G = D_5 \times \mathbb{Z}_{20}$. Similar to case above, we show that D_5 has 5 Sylow 2-subgroups and so has *G.*

Answer: there is no 3-Sylow subgroup, there is only one 5-Sylow subgroup and the number of 2-Sylow subgroup is one of numbers in the set {1, 5, 25}.