Answer on Question #81145 – Math – Linear Algebra

Question

Consider the linear operator $T\colon \mathbb{C}^3 \to \mathbb{C}^3$, defined by $T(z_1, z_2, z_3) =$
$= (z_1 - iz_2, iz_1 + 2z_2 + iz_3, -iz_2 + z_3)$.

i) Compute T* and check whether T is selfadjoint.
ii) Check whether T is unitary.

Solution

i) Given the rules
$z_1 \to z_1 - iz_2 = w_1$
$z_2 \to iz_1 + 2z_2 + iz_3 = w_2$
$z_3 \to -iz_2 + z_3 = w_3$
$w = Tz$
For linear operator T the matrix representation is

$$T = \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix}$$

We recall, that an operator T^* is called adjoint for the linear operator T if for all $x, y \in \mathbb{C}^3$ $(Tx, y) = (x, T^*y)$. The matrix representation for T^* can be found as

$$T^* = \left(\overline{\mathrm{T}}\right)^T = \overline{(T^T)}$$

where A^Tdenotes the transpose and $\overline{A}$ denotes the matrix with complex conjugated entries.
In our case

$$T^* = \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix} = T$$

The adjoint operator $T^*(z_1, z_2, z_3) = (z_1 - iz_2, iz_1 + 2z_2 + iz_3, -iz_2 + z_3)$.
Therefore, T is selfadjoint.

ii) A unitary operator is a bounded linear operator on a Hilbert space that satisfies $U^*U = UU^* = I$ where U^* is the adjoint of U.

$$T \cdot T^* = \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1(1) - i(i) + 0(0) & 1(-i) - i(2) + 0(-i) & 1(0) - i(i) + 0(1) \\ i(1) + 2(i) + i(0) & i(-i) + 2(2) + i(-i) & i(0) + 2(i) + i(1) \\ 0(1) - i(i) + 1(0) & 0(-i) - i(2) + 1(-i) & 0(0) - i(i) + 1(1) \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -3i & 1 \\ 3i & 6 & 3i \\ 1 & -3i & 2 \end{bmatrix} \neq I_3$$

Therefore, T is not unitary.

Answer:
i) T is selfadjoint.
ii) T is not unitary.