Answer on Question #81140 – Math – Differential Equations

Question

Solve

$$(2x+3)^2\frac{d^2y}{dx^2}-2(2x+3)\frac{dy}{dx}-12y=6x$$

Solution

Given differential equation is a Legendre's differential equation.

Let $2x+3=e^z \Rightarrow z=\ln(2x+3)$

$x=\frac{e^z-3}{2}$

$$\frac{dz}{dx}=\frac{2}{2x+3}$$

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{2}{2x+3}\cdot\frac{dy}{dz}$$

$(2x+3)\frac{dy}{dx}=2\frac{dy}{dz}=2Dy, D=\frac{d}{dz}$

$(2x+3)^2\frac{d^2y}{dx^2}=2^2D(D-1)y$

Substitute

$(2^2D(D-1)-2\cdot 2D-12)y=6\left(\frac{e^z-3}{2}\right)$

$(4D^2-4D-4D-12)y=3(e^z-3)$

$(4D^2-8D-12)y=3(e^z-3)$

We get a linear differential equation with constant coefficients.

$4D^2-8D-12=0$

$D^2-2D-3=0$

$D=\frac{2\pm\sqrt{(-2)^2-4(1)(-3)}}{2(1)}=1\pm 2$

$D=3,-1$

$y_C=c_1e^{3z}+c_2e^{-z}=c_1(2x+3)^3+c_2(2x+3)^{-1}$

General solution

$$y=y_C+Y_P$$

$Y_P=\frac{1}{4D^2-8D-12}\cdot 3(e^z-3)$

$=\frac{3}{4}\cdot\frac{1}{D^2-2D-3}e^z-\frac{9}{4}\cdot\frac{1}{D^2-2D-3}e^{0\cdot z}$

$=-\frac{3}{16}e^z+\frac{3}{4}=-\frac{3}{16}(2x+3)+\frac{3}{4}$

Therefore, general solution

$y=c_1(2x+3)^3+c_2(2x+3)^{-1}-\frac{3}{16}(2x+3)+\frac{3}{4}.$

Answer: $y = c_1(2x+3)^3 + c_2(2x+3)^{-1} - \frac{3}{16}(2x+3) + \frac{3}{4}$.