ANSWER on Question #81130 – Math – Linear Algebra

QUESTION

Which of the following are binary operations on $S = \{x \in \mathbb{R} | x > 0\}$? Justify your answer.

i) The operation Δ defined by $x\Delta y = |\ln(xy)|$ where $\ln(x)$ is the natural logarithm.

ii) The operation Δ defined by $x\Delta y = x^2 + y^3$. Also, for those operations which are binary operations, check whether they are associative or commutative.

SOLUTION

First, let us recall the definition of a binary operation.

A binary operation on a set S is a map which sends elements of the Cartesian product $S \times S$ to S:

$$\Delta : S \times S \to S$$

that is, a binary operation, it is an operation that takes any two elements x and y from the set S, does something with them (depends on the definition), and produces the third number c, which must be in the set S.

(More information: https://en.wikipedia.org/wiki/Binary_operation)

Next we recall the definition of associativity and commutativity.

A binary operation Δ on a set S is called associative if it satisfied the associative law:

$$(x\Delta y)\Delta z = x\Delta(y\Delta z), \quad \forall x, y, z \in S$$

(More information: https://en.wikipedia.org/wiki/Associative_property)

A binary operation Δ on a set S is called commutative if:

$$x\Delta y = y\Delta x, \quad \forall x, y \in S$$

(More information: https://en.wikipedia.org/wiki/Commutative_property)

Now we turn to the solution of the problem.

i)

$$x\Delta y = |\ln(xy)|$$

This is not a binary operation. If we choose $x = y = 1 > 0$, then,

$$1\Delta 1 = |\ln(1\cdot 1)| = |\ln(1)| = |0| = 0 \ngtr 0$$

Conclusion,

$$\boxed{x\Delta y = |\ln(xy)|\ is\ not\ a\ binary\ operation\ for\ \ S = \{x \in \mathbb{R}|x > 0\}}$$

ii)

$$x\Delta y = x^2 + y^3$$

As we know

$$x, y \in S \to \begin{cases} x > 0 \\ y > 0 \end{cases} \to \begin{cases} x^2 > 0 \\ y^3 > 0 \end{cases} \to \boxed{x^2 + y^3 > 0}$$

Conclusion,

$$\boxed{x\Delta y = x^2 + y^3\ is\ a\ binary\ operation\ for\ \ S = \{x \in \mathbb{R}|x > 0\}}$$

Associative:

$$(x\Delta y)\Delta z = (x\Delta y)^2 + z^3 = (x^2 + y^3)^2 + z^3 = \begin{bmatrix} We\ use\ the\ binomial\ formula \\ (a+b)^2 = a^2 + 2ab + b^2 \end{bmatrix} =$$

$$= ((x^2)^2 + 2\cdot x^2 \cdot y^3 + (y^3)^2) + z^3 = x^4 + 2x^2y^3 + y^6 + z^3$$

$$\boxed{(x\Delta y)\Delta z = x^4 + 2x^2y^3 + y^6 + z^3}$$

$$x\Delta(y\Delta z) = x^2 + (y\Delta z)^3 = x^2 + (y^2 + z^3)^3 = \begin{bmatrix} We\ use\ the\ binomial\ formula \\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \end{bmatrix} =$$

$$= x^2 + ((y^2)^3 + 3\cdot (y^2)^2 \cdot z^3 + 3\cdot y^2 \cdot (z^3)^2 + (z^3)^3) = x^2 + y^6 + 3y^4z^3 + 3y^2z^6 + z^9$$

$$\boxed{x\Delta(y\Delta z) = x^2 + y^6 + 3y^4z^3 + 3y^2z^6 + z^9}$$

Then,

$$\begin{cases} (x\Delta y)\Delta z = x^4 + 2x^2y^3 + y^6 + z^3 \\ x\Delta(y\Delta z) = x^2 + y^6 + 3y^4z^3 + 3y^2z^6 + z^9 \end{cases} \to \boxed{(x\Delta y)\Delta z \neq x\Delta(y\Delta z)}$$

Conclusion,

$$\boxed{x\Delta y = x^2 + y^3 \; is\ not\ an\ associative\ binary\ operation}$$

Commutative:

$$\begin{cases} x\Delta y = x^2 + y^3 \\ y\Delta x = y^2 + x^3 \end{cases} \to x\Delta y \neq y\Delta z$$

Conclusion,

$$\boxed{x\Delta y = x^2 + y^3 \; is\ not\ a\ commutative\ binary\ operation}$$

ANSWER

i)

$$x\Delta y = |\ln(xy)| \; is\ not\ a\ binary\ operation\ for\ \ S = \{x \in \mathbb{R} | x > 0\}$$

ii)

$$x\Delta y = x^2 + y^3 \; is\ a\ binary\ operation\ for\ \ S = \{x \in \mathbb{R} | x > 0\}$$

$$x\Delta y = x^2 + y^3 \; is\ not\ an\ associative\ binary\ operation$$

$x\Delta y = x^2 + y^3$ *is not a commutative binary operation.*