Answer on Question #81129 – Math– Linear Algebra

Question

Let $T: P^2 \to P^1$ be defined by $T(a + bx + cx^2) = b + 2c + (a - b)x$.
Check that T is a linear transformation.
Find the matrix of the transformation with respect to the ordered basis
$B_1 = \{x^2, x^2 + x, x^2 + x + 1\}$ and $B_2 = \{1, x\}$
Find the kernel of T.

Solution

A linear transformation between two vector spaces V and W is a map $T: V \to W$ such that the following properties hold:
(1) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for any vectors $v_1, v_2 \in V$.
(2) $T(\alpha \cdot v) = \alpha T(v)$ for any scalar α and any vector $v \in V$.
Check the properties in our case:
(1) Let $p(x) = k_0 + k_1 x + k_2 x^2$ and $q(x) = m_0 + m_1 x + m_2 x^2$ be polynomials in P^2.
$q(x) + p(x) = k_0 + k_1 x + k_2 x^2 + m_0 + m_1 x + m_2 x^2 =$
$= (k_0 + m_0) + (k_1 + m_1)x + (k_2 + m_2)x^2$
We get
$T(p + q) = T\big((k_0 + m_0) + (k_1 + m_1)x + (k_2 + m_2)x^2\big) =$
$= (k_1 + m_1) + 2(k_2 + m_2) + \big((k_0 + m_0) - (k_1 + m_1)\big)x =$
$= k_1 + m_1 + 2k_2 + 2m_2 + (k_0 + m_0 - k_1 - m_1)x$

$T(p) + T(q) = T(k_0 + k_1 x + k_2 x^2) + T(m_0 + m_1 x + m_2 x^2) =$
$= k_1 + 2k_2 + (k_0 - k_1)x + m_1 + 2m_2 + (m_0 - m_1)x =$
$= k_1 + m_1 + 2k_2 + 2m_2 + (k_0 + m_0 - k_1 - m_1)x$
Thus, we can see that

$$T(p + q) = T(p) + T(q),$$

so the property (1) holds.

(2) Let $p(x) = k_0 + k_1 x + k_2 x^2$ and let α be scalar.
$T(\alpha p) = T(\alpha k_0 + \alpha k_1 x + \alpha k_2 x^2) = \alpha k_1 + 2\alpha k_2 + (\alpha k_0 - \alpha k_1)x =$
$= \alpha(k_1 + 2k_2 + (k_0 - k_1)x)$
while
$\alpha T(p) = \alpha T(k_0 + k_1 x + k_2 x^2) = \alpha(k_1 + 2k_2 + (k_0 - k_1)x)$
Therefore,

$$T(\alpha p) = \alpha T(p),$$

so the property (2) holds.
Thus, T is a linear transformation.

To find the matrix of the transformation with respect to the ordered basis we need to apply the transformation to the basis and the result is the column of the transformation matrix.

$T(x^2) = T(0 + 0 \cdot x + 1 \cdot x^2) = 0 + 2 \cdot 1 + (0 - 0)x = 2$

$T(x^2 + x) = T(0 + 1 \cdot x + 1 \cdot x^2) = 1 + 2 \cdot 1 + (0 - 1)x = 3 - x$

$T(x^2 + x + 1) = T(1 + 1 \cdot x + 1 \cdot x^2) = 1 + 2 \cdot 1 + (1 - 1)x = 3$

Next, we find the coordinates for each of the above polynomials in the second basis:

$2 = 2 \cdot 1 + 0 \cdot x = (2, 0);$

$3 - x = 3 \cdot 1 + (-1) \cdot x = (3, -1);$

$3 = 3 \cdot 1 + 0 \cdot x = (3, 0).$

Thus, the matrix representation of T with respect to the given bases is

$$T[B_1, B_2] = \begin{bmatrix} 2 & 3 & 3 \\ 0 & -1 & 0 \end{bmatrix}$$

For transformation $T: P^2 \to P^1$the kernel (also called the null space is defined by

$$Ker(T) = \{p \in P^2 : T(p) = 0\}$$

so the kernel gives the elements from the original set P^2 that are mapped to zero by the transformation.

Let $p = k_0 + k_1 x + k_2 x^2$ is arbitrarily polynomial from P^2. Note that

$$T(p) = T(k_0 + k_1 x + k_2 x^2\ ) = k_1 + 2k_2 + (k_0 - k_1)x$$

Solve the equation

$$k_1 + 2k_2 + (k_0 - k_1)x = 0$$

Two polynomials are equal if and only if the coefficients of the corresponding powers are equal, hence we get the system of two equations:

$$\begin{cases} k_1 + 2k_2 = 0 \\ k_0 - k_1 = 0 \end{cases} => \begin{cases} k_1 = -2k_2 \\ k_0 = -2k_2 \end{cases}$$

Hence,

$$Ker(T) = \{p \in P^2 : p = -2k_2 - 2k_2 x + k_2 x^2\}.$$

Answer:

T is a linear transformation

$T[B_1, B_2] = \begin{bmatrix} 2 & 3 & 3 \\ 0 & -1 & 0 \end{bmatrix}$

$Ker(T) = \{p \in P^2 : p = -2k_2 - 2k_2 x + k_2 x^2\}.$