Answer on Question #81120 - Math - Abstract Algebra

Question. Give an example, with justification, of a group G with subgroups H and K such that HK is not a subgroup of G.

Answer. Choose G to be the symmetric group S_4. Let $h = (1\,2)$, $k = (1\,3\,4)$ be elements of G. Choose H and K to be the subgroups of G generated by h and k respectively. Clearly, $H = \{e, h\}$. We know that $h \in HK$, $k \in HK$, and $(kh)(2) = (k \circ h)(2) = 3$. As k fixes 2, every element of K fixes 2. Let $h' \in H$ and $k' \in K$. We have that $(h'k')(2) = (h' \circ k')(2) = h'(k'(2)) = h'(2)$ is either 1 or 2. Therefore, for all $h' \in H$ and $k' \in K$, $kh \neq h'k'$. In other words, $kh \notin HK$. This choice of h and k shows that HK is not closed under the operation of G. Hence HK is not a subgroup of G by the definition of subgroup.