Answer on Question #80967 – Math – Differential Equations

Question

2x+1)^2 d^2y/dx^2 - 2(2x+1) dy/dx- 12y= 6x
solve by taking substitution (2x+1)= e^t

Solution

Consider

$$2x+1=e^t$$

We can calculate x, t and the corresponding derivatives as

$$x=\frac{e^t-1}{2}$$

$$t=\log(2x+1)$$

$$\frac{dt}{dx}=\log(2x+1)'=\frac{2}{2x+1}$$

$$\frac{d^2t}{dx^2}=\log(2x+1)''=\frac{-4}{(2x+1)^2}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{2}{2x+1}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d^2t}{dx^2}\frac{dy}{dt}+\left(\frac{dt}{dx}\right)^2\frac{d^2y}{dt^2}=\frac{-4}{(2x+1)^2}\frac{dy}{dt}+\frac{4}{(2x+1)^2}\frac{d^2y}{dt^2}$$

Then the initial equation can be written as:

$$e^{2t}\left(\frac{-4}{e^{2t}}\frac{dy}{dt}+\frac{4}{e^{2t}}\frac{d^2y}{dt^2}\right)-2e^t\frac{2}{e^t}\frac{dy}{dt}-12y=3e^t-3$$

$$4\frac{d^2y}{dt^2}-8\frac{dy}{dt}-12y=3e^t-3$$

The general solution for this equation will be the sum of the complementary solution and a particular solution.
We find the complementary solution by solving

$$4\frac{d^2y}{dt^2}-8\frac{dy}{dt}-12y=0$$

Let's assume a solution will be proportional to e^{at} for some constant a

Substitute $y=e^{at}$ into the differential equation:

$$4\frac{d^2(e^{at})}{dt^2}-8\frac{d(e^{at})}{dt}-12(e^{at})=0$$

$$4a^2e^{at}-8ae^{at}-12e^{at}=0$$

Since $e^{at}\neq 0$ for any finite a, the zeros must come from the polynomial:

$$4a^2-8a-12=0$$

Factor

$$4(a-3)(a+1)=0$$

Thus, the roots are

$a_1=-1$

$a_2 = 3$

The root $a_1 = -1$ gives $y_1 = C_1 e^{-t}$ as a solution, where C_1 is an arbitrary constant.

The root $a_2 = 3$ gives $y_2 = C_2 e^{3t}$ as a solution, where C_2 is an arbitrary constant

The general solution is the sum of the above solutions:

$$y = C_1 e^{-t} + C_2 e^{3t}$$

We determine a particular solution to $4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 12y = 3e^t - 3$ by the method of undetermined coefficients:

A particular solution will be the sum of particular solutions to $4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 12y = -3$ and $4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 12y = 3e^t$

A particular solution to $4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 12y = -3$ is of the form $y_3 = b_1$

A particular solution to $4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 12y = 3e^t$ is of the form $y_4 = b_2 e^t$

Thus, the total particular solution can be found as

$$y_p = b_1 + b_2 e^t$$

Substituting

$$4\frac{d^2y_p}{dt^2} - 8\frac{dy_p}{dt} - 12y_p = 3e^t - 3$$

We get

$$4b_2e^t - 8b_2e^t - 12(\mathrm{b}_1 + b_2e^t) = 3e^t - 3$$

Simplifying

$$-12b_1 - 16b_2e^t = -3 + 3e^t$$

Equating the coefficients before the corresponding powers of e^t we get

$-12b_1 = -3$

$-16b_2 = 3$

So

$b_1 = 1/4$

$b_2 = \frac{-3}{16}$

Now substituting to y_p we get

$$y_p = \frac{1}{4} - \frac{3}{16}e^t$$

Now the general solution can be found as

$$y = C_1 e^{-t} + C_2 e^{3t} + \frac{1}{4} - \frac{3}{16}e^t$$

Now substituting $e^t = 2x + 1$ we get

$$y = \frac{C_1}{2x+1} + C_2(2x+1)^3 + \frac{1}{4} - \frac{3}{16}(2x+1)$$