ANSWER on Question #80929 – Math – Calculus

QUESTION

If

$$x \cdot \sin y = \sin(p + y),$$

show that

$$\sin p \cdot \frac{dy}{dx} + \sin^2 y = 0.$$

SOLUTION

We recall the rules for taking a derivative that we need to solve a given problem

$$1)\ \ \frac{dy}{dx} \equiv y'$$

$$2)\ \ f(x) = g(x) \to \frac{df}{dx} = \frac{dg}{dx}$$

$$3)\ \ y = f(x) \cdot g(x) \to \frac{dy}{dx} = f'(x) \cdot g(x) + f(x) \cdot g'(x) - \mathit{the\ product\ rule}$$

$$4)\ \ y = f\big(g(x)\big) \to \frac{dy}{dx} = f'\big(g(x)\big) \cdot g'(x) - \mathit{the\ chain\ rule}$$

$$5)\ \ y = \sin x \to \frac{dy}{dx} = \cos x$$

$$6)\ \ y = f(x + p) \to \frac{dy}{dx} = f'(x + p),\ \ \forall p \in \mathbb{R}$$

$$7)\ \ y = x \to \frac{dy}{dx} = (x)' = 1$$

(More information: https://en.wikipedia.org/wiki/Differentiation_rules)

In our case,

$$\frac{d}{dx}\cdot|x\cdot\sin y=\sin(p+y)\to\underbrace{\frac{d}{dx}(x\cdot\sin y)}_{the\ product\ rule}=\frac{d}{dx}(\sin(p+y))\to$$

$$(x)'\cdot\sin y+x\cdot\frac{d(\sin y)}{dx}=\cos(p+y)\cdot\frac{dy}{dx}\to 1\cdot\sin y+x\cdot\cos y\cdot\frac{dy}{dx}=\cos(p+y)\cdot\frac{dy}{dx}$$

$$\sin y+(x\cos y)\cdot\frac{dy}{dx}=\cos(p+y)\cdot\frac{dy}{dx}\to\sin y=\frac{dy}{dx}\cdot(\cos(p+y)-x\cos y)$$

According to the problem

$$x\cdot\sin y=\sin(p+y)\to\boxed{x=\frac{\sin(p+y)}{\sin y}}$$

Recall some formulas from trigonometry

$$1)\cos(p+y)=\cos p\cos y-\sin p\sin y$$

$$2)\quad\sin(p+y)=\sin p\cos y+\cos p\sin y$$

$$3)\quad\sin^2\alpha+\cos^2\alpha=1,\quad\forall\alpha\in\mathbb{R}$$

(More information: https://en.wikipedia.org/wiki/List_of_trigonometric_identities)

Then,

$$\sin y=\frac{dy}{dx}\cdot(\cos(y+p)-x\cos y)\to\sin y=\frac{dy}{dx}\cdot\left(\cos(p+y)-\frac{\sin(p+y)}{\sin y}\cdot\cos y\right)\to$$

$$\sin y=\frac{dy}{dx}\cdot\left(\frac{\cos(p+y)\cdot\sin y-\sin(p+y)\cdot\cos y}{\sin y}\right)$$

We transform the expression in parentheses

$$\frac{\cos(p+y)\cdot\sin y - \sin(p+y)\cdot\cos y}{\sin y} =$$

$$= \frac{(\cos p\cos y - \sin p\sin y)\cdot\sin y - (\sin p\cos y + \cos p\sin y)\cdot\cos y}{\sin y} =$$

$$= \frac{\cos p\cos y\sin y - \sin p\sin^2 y - \sin p\cos^2 y - \cos p\cos y\sin y}{\sin y} =$$

$$= \frac{-\sin p\cdot\overbrace{(\sin^2 y + \cos^2 y)}^{=1}}{\sin y} = \frac{-\sin p\cdot 1}{\sin y} = \frac{-\sin p}{\sin y}$$

By collecting all the formulas obtained in one, we obtain

$$\sin y = \frac{dy}{dx}\cdot\left(\frac{\cos(p+y)\cdot\sin y - \sin(p+y)\cdot\cos y}{\sin y}\right) \to \sin y = \frac{dy}{dx}\cdot\left(\frac{-\sin p}{\sin y}\right) \to$$

$$\sin y = \frac{dy}{dx}\cdot\left(\frac{-\sin p}{\sin y}\right)\Bigg| \times (\sin y) \to \sin^2 y = \sin y\cdot\frac{-\sin p}{\sin y}\cdot\frac{dy}{dx} \to \sin^2 y = -\sin p\cdot\frac{dy}{dx} \to$$

$$\boxed{\sin^2 y + \sin p\cdot\frac{dy}{dx} = 0 \leftrightarrow \sin p\cdot\frac{dy}{dx} + \sin^2 y = 0}$$

Q.E.D.