Answer to Question #80892 - Math / Abstract Algebra

Question. Let p be a prime and $a \in \mathbb{N}$ such that $p \mid a^{50}$. Show that $p^{50} \mid a^{50}$.

Answer. We will prove a more general statement: for every non-zero $n \in \mathbb{N}$, if $p \mid a^n$, then $p^n \mid a^n$. Applying this statement with $n = 50$ gives the desired statement. This more general statement will be proved in two stages as follows.

- We will prove that if $p \mid a^n$, then $p \mid a$. The proof is by induction.
  - Induction base with $n = 1$. Assume that $p \mid a^1$. It is the same as $p \mid a$ because $a^1 = a$.
  - Induction step. Let $n \in \mathbb{N}$ be non-zero. Assume that $p \mid a^n$ implies $p \mid a$. Assume that $p \mid a^{n+1}$. We have that $p \mid a^n a$, then as p is prime, by Euclid's lemma, $p \mid a^n$ or $p \mid a$.
    - If $p \mid a^n$, then $p \mid a$ by the induction hypothesis.
    - If $p \mid a$, there is nothing to prove.

    We considered all cases, and $p \mid a$ is true in all cases.
- We will prove that if $p \mid a$, then $p^n \mid a^n$. Assume that $p \mid a$. The proof is by induction.
  - Induction base with $n = 1$. We have $p^1 \mid a^1$ by the assumption because $a^1 = a$ and $p^1 = p$.
  - Induction step. Let $n \in \mathbb{N}$ be non-zero. Assume that $p^n \mid a^n$. By the properties of division, the induction hypothesis, and the assumption $p \mid a$, we have $p^n p \mid a^n a$. This is the same as $p^{n+1} \mid a^{n+1}$.