Answer on Question #80861 – Math – Linear Algebra

Question

Find the inverse of the matrix $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ using row reduction.

Solution

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Form the augmented matrix $[A \quad | \quad I]$

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{(-1)\cdot R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1+(2)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_3-(2)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4 & -1 \\ 0 & 1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

As can be seen, we have obtained the identity matrix to the left. So, we are done.

$$A^{-1} = \begin{bmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Indeed,

$$AA^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(-2) + 2(-1) + 1(1) & -1(4) + 2(3) + 1(-2) & -1(-1) + 2(-1) + 1(1) \\ 0(-2) + 1(-1) + 1(1) & 0(4) + 1(3) + 1(-2) & 0(-1) + 1(-1) + 1(1) \\ 1(-2) + 0(-1) + 2(1) & 1(4) + 0(3) + 2(-2) & 1(-1) + 0(-1) + 2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Answer:

$$A^{-1} = \begin{bmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$$