Answer on Question #80859 – Math – Linear Algebra

Question

Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, x) = (x + y, y, 2x - 2y + 2z)$. Check that T satisfies the polynomial $(x - 1)^2(x - 2)$. Find the minimal polynomial of T.

Solution

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix}$$

Find $(T - I)^2(T - 2I)$

$$T - I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$(T - I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0(0) + 1(0) + 0(2) & 0(1) + 1(0) + 0(-2) & 0(0) + 1(0) + 0(1) \\ 0(0) + 0(0) + 0(2) & 0(1) + 0(0) + 0(-2) & 0(0) + 0(0) + 0(1) \\ 2(0) - 2(0) + 1(2) & 2(1) - 2(0) + 1(-2) & 2(0) - 2(0) + 1(1) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$T - 2I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$$(T - I)^2(T - 2I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0(-1) + 0(0) + 0(2) & 0(1) + 0(-1) + 0(-2) & 0(0) + 0(0) + 0(0) \\ 0(-1) + 0(0) + 0(2) & 0(1) + 0(-1) + 0(-2) & 0(0) + 0(0) + 0(0) \\ 2(-1) + 0(0) + 1(2) & 2(1) + 0(-1) + 1(-2) & 2(0) + 0(0) + 1(0) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the characteristic polynomial is obtained as
$p(\lambda) = \det(T - \lambda I) = (\lambda - 1)^2(\lambda - 2)$

$$(T-I)(T-2I)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 2 & -2 & 1\end{bmatrix}\cdot\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 2 & -2 & 0\end{bmatrix}=$$

$$=\begin{bmatrix}0(-1)+1(0)+0(2) & 0(1)+1(-1)+0(-2) & 0(0)+1(0)+0(0)\\ 0(-1)+0(0)+0(2) & 0(1)+0(-1)+0(-2) & 0(0)+0(0)+0(0)\\ 2(-1)-2(0)+1(2) & 2(1)-2(-1)+1(-2) & 2(0)-2(0)+1(0)\end{bmatrix}=$$

$$=\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 2 & 0\end{bmatrix}\neq\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The minimal polynomial of T is

$$m_T(x)=(x-1)^2(x-2)$$

Answer: $m_T(x)=(x-1)^2(x-2)$.