Answer on Question #80854 – Math – Linear Algebra

Question

Find the orthogonal canonical reduction of the quadratic form x2+y2+z2−2xy−2xz−2yz. Also,find its principal axes.

Solution

$f(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$

The matrix of the quadratic form is

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$

hence

$(1-\lambda)\big((1-\lambda)^2 - 1\big) + (-1 + \lambda - 1) - (1 + 1 - \lambda) = 0$

$(1-\lambda)\left(\lambda^2 - 2\lambda\right) - 4 + 2\lambda = 0$

$(\lambda - 2)^2(\lambda + 1) = 0$

$\lambda_1 = 2, \lambda_2 = -1.$

Thus, the orthogonal canonical reduction is

$$Q = (x' \quad y' \quad z') \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 2(x')^2 + 2(y')^2 - (z')^2$$

Find eigenvectors:

$\lambda = 2$

$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$x + y + z = 0$

Orthogonal solutions to this equation are $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. These normed vectors are principal axes $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

$\lambda = -1$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{cases} 2x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + 2z = 0 \end{cases}$$

$$\begin{cases} z = 2x - y \\ -x + 2y - 2x + y = 0 \\ -x - y + 4x - 2y = 0 \end{cases}$$

$$\begin{cases} z = 2x - y \\ y = x \\ y = x \end{cases}$$

The normed solution is principal axis $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.