Question 1. *If* (a_n) *and* (b_n) *are Cauchy sequences, show directly that* $(a_n + b_n)$ *is also a Cauchy sequence.*

Solution. Since (a_n) and (b_n) are Cauchy sequences, for any $\varepsilon > 0$ there are $N_1 \in \mathbb{N}$, such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $m, n > N_1$, and $N_2 \in \mathbb{N}$, such that $|b_n - b_m| < \frac{\varepsilon}{2}$ for all $m, n > N_2$. Set $N = \max\{N_1, N_2\}$. Then for all $m, n > N$ we have

$$|(a_n+b_n)-(a_m+b_m)| = |a_n-a_m+b_n-b_m| \leq |a_n-a_m|+|b_n-b_m| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(a_n + b_n)$ is a Cauchy sequence. □