ANSWER on Question #80494 – Math – Differential Equations

QUESTION

Using the method of undermined coefficient find the general solution of D.E

$$y^{(IV)} - 2y''' + 2y'' = 3ye^{-x} + 2e^{-x} \cdot x + e^{-x} \cdot \sin x$$

SOLUTION

Hint: In the solution, I will proceed from the assumption that the question contained a mistake and the equation actually looks like this:

$$y^{(IV)} - 2y''' + 2y'' = 3e^{-x} + 2e^{-x} \cdot x + e^{-x} \cdot \sin x$$

As we know from theory, the solution of a differential equation of the form

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = f_1(x) + f_2(x) + f_3(x) + \cdots f_n(x)$$

consists of such parts:

1) solution of the homogeneous equation

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0 \rightarrow y = y_h(x)$$

2) partial solutions of inhomogeneous equations

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = f_1(x) \rightarrow y = y_1(x)$$

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = f_2(x) \rightarrow y = y_2(x)$$

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = f_3(x) \rightarrow y = y_3(x)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = f_n(x) \rightarrow y = y_n(x)$$

Then,

$$y_{sol} = y_h(x) + y_1(x) + y_2(x) + y_3(x) + \cdots + y_n(x)$$

In our case,

$$y^{(IV)} - 2y''' + 2y'' = 3e^{-x} + 2e^{-x} \cdot x + e^{-x} \cdot \sin x$$

1 STEP: We find the solution of the homogeneous equation

$$y^{(IV)} - 2y''' + 2y'' = 0$$

The solution will be sought in the form

$$y = e^{kx} \to v \to$$

$$y^{(IV)} - 2y''' + 2y'' = 0 \to k^4 \cdot e^{kx} - 2k^3 \cdot e^{kx} + 2k^2 \cdot e^{kx} = 0 \to e^{kx} \cdot (k^4 - 2k^3 + 2k^2) = 0 \to$$

$$k^4 - 2k^3 + 2k^2 = 0 \to k^2 \cdot (k^2 - 2k + 2) = 0 \to \left[\begin{array}{c} k^2 = 0 \\ k^2 - 2k + 2 = 0 \end{array}\right.$$

$$k^2 = 0 \to \boxed{k_{1,2} = 0 - multiple\ root}$$

$$k^2 - 2k + 2 = 0 \to \begin{cases} a = 1 \\ b = -2 \\ c = 2 \end{cases} \to D = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot 2 = 4 - 8 = -4 \to$$

$$\sqrt{D} = \sqrt{-4} = 2i \to \left[\begin{array}{l} k_3 = \dfrac{-b - \sqrt{D}}{2a} = \dfrac{-(-2) - 2i}{2 \cdot 1} = \dfrac{2 - 2i}{2} = 1 - i \\ k_4 = \dfrac{-b + \sqrt{D}}{2a} = \dfrac{-(-2) + 2i}{2 \cdot 1} = \dfrac{2 + 2i}{2} = 1 + i \end{array}\right.$$

Then,

$$y_h(x) = \underbrace{e^{k_1 x}(A + Bx)}_{since\ the\ root\ of\ multiplicity\ 2} + C_1 e^{k_3 x} + C_2 e^{k_4 x} \to$$

$$y_h(x) = e^{0 \cdot x}(A + Bx) + C_1 e^{(1+i)x} + C_3 e^{(1-i)x} \equiv A + Bx + A_1 e^x \cdot \cos x + A_2 e^x \cdot \sin x$$

$$Hint: \begin{cases} e^{ix} = \cos x + i \cdot \sin x \\ e^{-ix} = \cos x + i \cdot \sin x \end{cases} \to C_1 e^{ix} + C_2 e^{-ix} \equiv A_1 \cos x + A_2 \sin x$$

Conclusion,

$$\boxed{y^{(IV)} - 2y''' + 2y'' = 0 \to y_h(x) = A + Bx + A_1 e^x \cdot \cos x + A_2 e^x \cdot \sin x}$$

2 STEP: We find a particular solution of the inhomogeneous equations

a) $f_1(x) = 3e^{-x}$

$$y^{(IV)} - 2y''' + 2y'' = 3e^{-x}$$

The solution will be sought in the form

$$y(x) = Ce^{-x} \to \begin{cases} y^{(IV)} = C \cdot (-1)^4 \cdot e^{-x} = Ce^{-x} \\ y''' = C \cdot (-1)^3 \cdot e^{-x} = -Ce^{-x} \\ y'' = C \cdot (-1)^2 \cdot e^{-x} = Ce^{-x} \end{cases}$$

Then,

$$y^{(IV)} - 2y''' + 2y'' = 3e^{-x} \to Ce^{-x} - 2 \cdot (-Ce^{-x}) + 2 \cdot Ce^{-x} = 3e^{-x} \to 5Ce^{-x} = 3e^{-x} | \div (5e^{-x}) \to$$

$$C = \frac{3e^{-x}}{5e^{-x}} = \frac{3}{5}$$

Conclusion,

$$\boxed{y^{(IV)} - 2y''' + 2y'' = 3e^{-x} \to y_1(x) = \frac{3}{5}e^{-x}}$$

b) $f_2(x) = 2e^{-x} \cdot x$

$$y^{(IV)} - 2y''' + 2y'' = 2e^{-x} \cdot x$$

The solution will be sought in the form

$$y(x) = e^{-x} \cdot (Ax + B) \to \begin{cases} y' = -e^{-x} \cdot (Ax + B) + Ae^{-x} \equiv e^{-x} \cdot (A - B - Ax) \\ y'' = -e^{-x} \cdot (A - B - Ax) - Ae^{-x} \equiv e^{-x} \cdot (B - 2A + Ax) \\ y''' = -e^{-x} \cdot (B - 2A + Ax) + Ae^{-x} \equiv e^{-x} \cdot (3A - B - Ax) \\ y^{(IV)} = -e^{-x} \cdot (3A - B - Ax) - Ae^{-x} \equiv e^{-x} \cdot (B - 4A + Ax) \end{cases}$$

Then,

$$y^{(IV)} - 2y''' + 2y'' = 2e^{-x} \cdot x \to$$

$$e^{-x} \cdot (B - 4A + Ax) - 2 \cdot e^{-x} \cdot (3A - B - Ax) + 2 \cdot e^{-x} \cdot (B - 2A + Ax) = e^{-x} \cdot (2x) \to$$

$$e^{-x} \cdot (B - 4A + Ax - 6A + 2B + 2Ax + 2B - 4A + 2Ax) = e^{-x} \cdot (2x) \to$$

$$e^{-x} \cdot (5B - 14A + 5Ax) = e^{-x} \cdot (2x) \to 5B - 14A + 5Ax = 2x \to \begin{cases} 5A = 2 | \div (5) \\ 5B - 14A = 0 \end{cases} \to$$

$$\begin{cases} A = \frac{2}{5} \\ 5B = 14 \cdot \frac{2}{5} \Big| \div (5) \end{cases} \to \boxed{\begin{cases} A = \frac{2}{5} \\ B = \frac{28}{25} \end{cases}}$$

Conclusion,

$$\boxed{y^{(IV)} - 2y''' + 2y'' = 2e^{-x} \cdot x \rightarrow y_2(x) = e^{-x} \cdot \left(\frac{2x}{5} + \frac{28}{25}\right)}$$

c) $f_3(x) = e^{-x} \cdot \sin x$

$$y^{(IV)} - 2y''' + 2y'' = e^{-x} \cdot \sin x$$

The solution will be sought in the form

$$y(x) = Ae^{-x} \cdot \sin x + Be^{-x} \cdot \cos x \rightarrow$$

$$y' = -Ae^{-x} \cdot \sin x + Ae^{-x} \cdot \cos x - Be^{-x} \cdot \cos x - Be^{-x} \cdot \sin x \rightarrow$$

$$\boxed{y' = (-A - B)e^{-x} \cdot \sin x + (A - B)e^{-x} \cdot \cos x}$$

$$y'' = -(-A - B)e^{-x} \cdot \sin x + (-A - B)e^{-x} \cdot \cos x - (A - B)e^{-x} \cdot \cos x - (A - B)e^{-x} \cdot \sin x \rightarrow$$

$$y'' = (A + B - A + B)e^{-x} \cdot \sin x + (-A - B - A + B)e^{-x} \cdot \cos x \rightarrow$$

$$\boxed{y'' = (2B)e^{-x} \cdot \sin x + (-2A)e^{-x} \cdot \cos x}$$

$$y''' = -(2B)e^{-x} \cdot \sin x + (2B)e^{-x} \cdot \cos x - (-2A)e^{-x} \cdot \cos x - (-2A)e^{-x} \cdot \sin x \rightarrow$$

$$\boxed{y''' = (-2B + 2A)e^{-x} \cdot \sin x + (2B + 2A)e^{-x} \cdot \cos x}$$

$$y'''' = -(-2B + 2A)e^{-x} \cdot \sin x + (-2B + 2A)e^{-x} \cdot \cos x - (2B + 2A)e^{-x} \cdot \cos x - (2B + 2A)e^{-x} \cdot \sin x$$

$$y'''' = (2B - 2A - 2B - 2A)e^{-x} \cdot \sin x + (-2B + 2A - 2B - 2A)e^{-x} \cdot \cos x \rightarrow$$

$$\boxed{y'''' = (-4A)e^{-x} \cdot \sin x + (-4B)e^{-x} \cdot \cos x}$$

Then,

$$y^{(IV)} - 2y''' + 2y'' = 2e^{-x} \cdot \sin x \rightarrow$$

$$(-4A)e^{-x} \cdot \sin x + (-4B)e^{-x} \cdot \cos x - 2\big((-2B + 2A)e^{-x} \cdot \sin x + (2B + 2A)e^{-x} \cdot \cos x\big) +$$

$$+2\big((2B)e^{-x} \cdot \sin x + (-2A)e^{-x} \cdot \cos x\big) = e^{-x} \cdot \sin x \rightarrow$$

$$(-4A)e^{-x} \cdot \sin x + (-4B)e^{-x} \cdot \cos x + (4B - 4A)e^{-x} \cdot \sin x + (-4B - 4A)e^{-x} \cdot \cos x +$$

$$+(4B)e^{-x} \cdot \sin x + (-4A)e^{-x} \cdot \cos x = e^{-x} \cdot \sin x \rightarrow$$

$$(-4A + 4B - 4A + 4B)e^{-x} \cdot \sin x + (-4B - 4B - 4A - 4A)e^{-x} \cdot \cos x = e^{-x} \cdot \sin x \rightarrow$$

$$(-8A + 8B)e^{-x} \cdot \sin x + (-8B - 8A)e^{-x} \cdot \cos x = e^{-x} \cdot \sin x \rightarrow$$

$$\begin{cases}-8A+8B=1\\-8A-8B=0\end{cases}\rightarrow\begin{cases}-8A+8B=1\\-8A=8B\mid\div(-8)\end{cases}\rightarrow\begin{cases}-8\cdot(-B)+8B=1\\A=-B\end{cases}\rightarrow\begin{cases}16B=1\\A=-B\end{cases}\rightarrow$$

$$\boxed{\begin{cases}A=-\dfrac{1}{16}\\B=\dfrac{1}{16}\end{cases}}$$

Conclusion,

$$\boxed{y^{(IV)}-2y'''+2y''=2e^{-x}\cdot\sin x\rightarrow y_3(x)=-\frac{\sin x}{16}\cdot e^{-x}+\frac{\cos x}{16}\cdot e^{-x}}$$

General conclusion,

$$y^{(IV)}-2y'''+2y''=3ye^{-x}+2e^{-x}\cdot x+e^{-x}\cdot\sin x\rightarrow$$

$$y(x)=y_h(x)+y_1(x)+y_2(x)+y_3(x)\rightarrow$$

$$y(x)=A+Bx+A_1e^x\cdot\cos x+A_2e^x\cdot\sin x+\frac{3}{5}e^{-x}+e^{-x}\cdot\left(\frac{2x}{5}+\frac{28}{25}\right)-\frac{\sin x}{16}\cdot e^{-x}+\frac{\cos x}{16}\cdot e^{-x}\rightarrow$$

$$y(x)=A+Bx+A_1e^x\cdot\cos x+A_2e^x\cdot\sin x+\left(\frac{3}{5}+\frac{28}{25}\right)\cdot e^{-x}+\frac{2x}{5}\cdot e^{-x}-\frac{\sin x}{16}\cdot e^{-x}+\frac{\cos x}{16}\cdot e^{-x}\rightarrow$$

$$\boxed{y(x)=A+Bx+A_1e^x\cdot\cos x+A_2e^x\cdot\sin x+\frac{43}{25}\cdot e^{-x}+\frac{2x}{5}\cdot e^{-x}-\frac{\sin x}{16}\cdot e^{-x}+\frac{\cos x}{16}\cdot e^{-x}}$$

ANSWER

$$y(x)=A+Bx+A_1e^x\cdot\cos x+A_2e^x\cdot\sin x+\frac{43}{25}\cdot e^{-x}+\frac{2x}{5}\cdot e^{-x}-\frac{\sin x}{16}\cdot e^{-x}+\frac{\cos x}{16}\cdot e^{-x}$$