Answer on Question #80410 – Math – Trigonometry

Question

If $\cos^3\theta + 3\cos\theta\sin^2\theta = x, \sin^3\theta + 3\sin\theta\cos^2\theta = y,$ then prove

$$(x+y)^{2/3} + (x-y)^{2/3} = 2$$

Solution

$x + y = \cos^3\theta + 3\cos\theta\sin^2\theta + \sin^3\theta + 3\sin\theta\cos^2\theta$
$= \cos^3\theta + \sin^3\theta + 3\sin\theta\cos\theta(\cos\theta + \sin\theta)$
$= (\cos\theta + \sin\theta)(\cos^2\theta - \sin\theta\cos\theta + \sin^2\theta) + 3\sin\theta\cos\theta(\cos\theta + \sin\theta)$
$= (\cos\theta + \sin\theta)(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta) = (\cos\theta + \sin\theta)^3$

$x - y = \cos^3\theta + 3\cos\theta\sin^2\theta - \sin^3\theta - 3\sin\theta\cos^2\theta$
$= \cos^3\theta - \sin^3\theta + 3\sin\theta\cos\theta(\sin\theta - \cos\theta)$
$= (\cos\theta - \sin\theta)(\cos^2\theta + \sin\theta\cos\theta + \sin^2\theta) - 3\sin\theta\cos\theta(\cos\theta - \sin\theta)$
$= (\cos\theta - \sin\theta)(\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta) = (\cos\theta - \sin\theta)^3$

$(x+y)^{2/3} + (x-y)^{2/3} = ((\cos\theta + \sin\theta)^3)^{2/3} + ((\cos\theta - \sin\theta)^3)^{2/3}$
$= (\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2$
$= \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta$
$= 2(\cos^2\theta + \sin^2\theta) = 2(1) = 2, \theta \in R.$