ANSWER on Question #80273 – Math – Differential Equations

QUESTION

Solve the PDE:

1)

$$(D^2 + D - 1)z = 4e^{x+y}\cos(x+y)$$

2)

$$(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$$

SOLUTION

Before solving the task, let us recall some theoretical facts.

Let the given differential equation be

$$F(D, D') = f(x, y).$$

Factorize $F(D, D')$ into linear factors. Then use the following results:

Rule I. Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of C.F. is taken as

$$e^{\left(\frac{cx}{b}\right)}\varphi(by + ax), \;\; if\ b \neq 0$$

We now have three particular cases of Rule I:

Rule IA. Take $c = 0$ in Rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is

$$\varphi(by + ax), \;\; if\ b \neq 0.$$

Rule IB. Take $a = 0$ in Rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is

$$e^{\left(\frac{cx}{b}\right)}\varphi(by), \;\; if\ b \neq 0.$$

Rule IC. Take $a = c = 0$ and $b = 1$ in Rule I. Hence corresponding to each linear factor $(1 \cdot D)$, the part of C.F. is

$$\varphi(y).$$

Rule II. When

$$f(x,y) = V(x,y)e^{ax+by}$$

Then,

$$P.I. = \frac{1}{F(D,D')} V e^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} V(x,y)$$

In our case,

1)

$$(D^2 + D - 1)z = 4e^{x+y}\cos(x+y)$$

$$(D^2 + D - 1)z = 4e^{x+y}\cos(x+y) \to z(x,y) = C.F. + P.I.$$

0 STEP: Factorize $F(D,D')$ into linear factors.

$$D^2 + D - 1 \to \underbrace{1}_{a} \cdot x^2 + \underbrace{1}_{b} \cdot x \underbrace{-1}_{c} = 0 \to \sqrt{D} = \sqrt{b^2 - 4ac} = \sqrt{(1)^2 - 4 \cdot 1 \cdot (-1)} = \sqrt{1+4}$$

$$\sqrt{D} = \sqrt{5} \to \begin{cases} m_1 = \dfrac{-b - \sqrt{D}}{2a} = \dfrac{-1 - \sqrt{5}}{2 \cdot 1} = -\dfrac{1 + \sqrt{5}}{2} \\ m_2 = \dfrac{-b + \sqrt{D}}{2a} = \dfrac{-1 + \sqrt{5}}{2 \cdot 1} = -\dfrac{1 - \sqrt{5}}{2} \end{cases}$$

Conclusion,

$$m^2 + m - 1 = (m - m_1)(m - m_2) = \left(m + \frac{1+\sqrt{5}}{2}\right)\left(m + \frac{1-\sqrt{5}}{2}\right) \to$$

$$D^2 + D - 1 = \left(D + \frac{1+\sqrt{5}}{2}\right)\left(D + \frac{1-\sqrt{5}}{2}\right)$$

Then,

$$\boxed{(D^2 + D - 1)z = 4e^{x+y}\cos(x+y) \to \left(D + \frac{1+\sqrt{5}}{2}\right)\left(D + \frac{1-\sqrt{5}}{2}\right) z = 4e^{x+y}\cos(x+y)}$$

1 STEP: Let find C.F.

$$\begin{cases}\left(D+\frac{1+\sqrt{5}}{2}\right)z \\ (bD-aD'-c)z\end{cases} \to \begin{cases} b=1 \\ a=0 \\ c=-\frac{1+\sqrt{5}}{2}\end{cases} \to (C.F.)_1 = e^{\left(\frac{-\frac{1+\sqrt{5}}{2}\cdot x}{1}\right)} \cdot \varphi_1(1\cdot y+(0)\cdot x) \to$$

$$\boxed{(C.F.)_1 = e^{-\frac{1+\sqrt{5}}{2}\cdot x} \cdot \varphi_1(y), where\ \varphi_1\ is\ arbitrary\ function}$$

$$\begin{cases}\left(D+\frac{1-\sqrt{5}}{2}\right)z \\ (bD-aD'-c)z\end{cases} \to \begin{cases} b=1 \\ a=0 \\ c=-\frac{1-\sqrt{5}}{2}\end{cases} \to (C.F.)_2 = e^{\left(\frac{-\frac{1-\sqrt{5}}{2}\cdot x}{1}\right)} \cdot \varphi_2(1\cdot y+(0)\cdot x) \to$$

$$\boxed{(C.F.)_2 = e^{-\frac{1-\sqrt{5}}{2}x} \cdot \varphi_2(y), where\ \varphi_2\ is\ arbitrary\ function}$$

Then,

$$C.F. = (C.F.)_1 + (C.F.)_2 \to \boxed{C.F. = e^{-\frac{1+\sqrt{5}}{2}\cdot x}\cdot\varphi_1(y) + e^{-\frac{1-\sqrt{5}}{2}x}\cdot\varphi_2(y)}$$

2 STEP: Let find P.I.

$$P.I. = \frac{1}{D^2+D-1}4e^{\overset{a}{\widehat{1}}\cdot x+\overset{b}{\widehat{1}}\cdot y}\cos(x+y) = 4e^{x+y}\frac{1}{(D+1)^2+(D+1)-1}\cos(x+y) =$$

$$= 4e^{x+y}\frac{1}{D^2+2D+1+D+1-1}\cos(x+y) = 4e^{x+y}\frac{1}{D^2+3D+1}\cos(x+y) =$$

$$= 4e^{x+y}\frac{1}{(-1)^2+3D+1}\cos(x+y) = 4e^{x+y}\frac{1}{1+3D+1}\cos(x+y) =$$

$$= 4e^{x+y}\frac{1}{3D+2}\cos(x+y) = 4e^{x+y}\frac{(3D-2)}{(3D+2)(3D-2)}\cos(x+y) =$$

$$= 4e^{x+y}\frac{(3D-2)}{9D^2-4}\cos(x+y) = 4e^{x+y}\frac{(3D-2)}{9\cdot(-1)^2-4}\cos(x+y) == 4e^{x+y}\frac{(3D-2)}{9\cdot 1-4}\cos(x+y)$$

$$= 4e^{x+y}\frac{(3D-2)}{5}\cos(x+y) =$$

$$= \frac{4}{5}e^{x+y}(3D-2)\cos(x+y) = \frac{4}{5}e^{x+y}\left(3\frac{\partial}{\partial x}-2\right)\cos(x+y) =$$

$$= \frac{4}{5}e^{x+y}\left(3\frac{\partial}{\partial x}(\cos(x+y)) - 2\cdot\cos(x+y)\right) =$$

$$= \frac{4}{5} e^{x+y} (3(-\sin(x+y)) - 2\cos(x+y)) =$$

$$= -\frac{4}{5} e^{x+y} (3\sin(x+y) + 2\cos(x+y))$$

Then,

$$\boxed{P.I. = -\frac{4}{5} e^{x+y} (3\sin(x+y) + 2\cos(x+y))}$$

Conclusion,

$$z(x,y) = C.F. + P.I. =$$

$$= e^{-\frac{1+\sqrt{5}}{2} \cdot x} \cdot \varphi_1(y) + e^{-\frac{1-\sqrt{5}}{2} x} \cdot \varphi_2(y) - \frac{4}{5} e^{x+y} (3\sin(x+y) + 2\cos(x+y))$$

$$z(x,y) = e^{-\frac{1+\sqrt{5}}{2} \cdot x} \cdot \varphi_1(y) + e^{-\frac{1-\sqrt{5}}{2} x} \cdot \varphi_2(y) - \frac{4}{5} e^{x+y} (3\sin(x+y) + 2\cos(x+y))$$

$$\boxed{\begin{cases} z(x,y) = e^{-\frac{1+\sqrt{5}}{2} \cdot x} \cdot \varphi_1(y) + e^{-\frac{1-\sqrt{5}}{2} x} \cdot \varphi_2(y) - \frac{4}{5} e^{x+y} (3\sin(x+y) + 2\cos(x+y)) \\ \text{where } \varphi_1 \text{and } \varphi_2 \text{ are arbitrary functions} \end{cases}}$$

2)

$$(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$$

1 STEP: Let find C.F.

$$\begin{cases} (D + D' - 1)z \\ (bD - aD' - c)z \end{cases} \rightarrow \begin{cases} b = 1 \\ a = -1 \\ c = 1 \end{cases} \rightarrow (C.F.)_1 = e^{\left(\frac{1\cdot x}{1}\right)} \cdot \varphi_1(1 \cdot y + (-1) \cdot x) \rightarrow$$

$$\boxed{(C.F.)_1 = e^x \cdot \varphi_1(y - x), where\ \varphi_1\ is\ arbitrary\ function}$$

$$\begin{cases} (D + 2D' - 3)z \\ (bD - aD' - c)z \end{cases} \rightarrow \begin{cases} b = 1 \\ a = -2 \\ c = 3 \end{cases} \rightarrow (C.F.)_2 = e^{\left(\frac{3\cdot x}{1}\right)} \cdot \varphi_2(1 \cdot y + (-2) \cdot x) \rightarrow$$

$$\boxed{(C.F.)_2 = e^{3x} \cdot \varphi_2(y - 2x), where\ \varphi_2\ is\ arbitrary\ function}$$

Then,

$$C.F. = (C.F.)_1 + (C.F.)_2 \rightarrow \boxed{C.F. = e^x \cdot \varphi_1(y - x) + e^{3x} \cdot \varphi_2(y - 2x)}$$

2 STEP: Let find P.I.

$$P.I. = \frac{1}{(D + D' - 1)(D + 2D' - 3)}(4 + 3x + 6y) =$$

$$= \frac{1}{(-1) \cdot (1 - [D + D']) \cdot (-3) \cdot \left(1 - \left[\frac{D + 2D'}{3}\right]\right)}(4 + 3x + 6y) =$$

$$= \frac{1}{3} \cdot (1 - [D + D'])^{-1} \cdot \left(1 - \left[\frac{D + 2D'}{3}\right]\right)^{-1}(4 + 3x + 6y) =$$

$$= \left[\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots, |x| < 1\right] =$$

$$= \frac{1}{3} \cdot (1 + [D + D'] + [D + D']^2 + \cdots) \cdot \left(1 + \left[\frac{D + 2D'}{3}\right] + \left[\frac{D + 2D'}{3}\right]^2 + \cdots\right)(4 + 3x + 6y)$$

$$= \frac{1}{3} \cdot \left(1 + [D + D'] + \left[\frac{D + 2D'}{3}\right] + \cdots\right)(4 + 3x + 6y) =$$

$$= \frac{1}{3} \cdot \left(1 + D + D' + \frac{D}{3} + \frac{2D'}{3}\right)(4 + 3x + 6y) = \frac{1}{3}\left(1 + \frac{4D}{3} + \frac{5D'}{3}\right)(4 + 3x + 6y) =$$

$$=\frac{1}{3}\cdot\left(4+3x+6y+\frac{4}{3}\cdot\frac{\partial}{\partial x}(4+3x+6y)+\frac{5}{3}\cdot\frac{\partial}{\partial y}(4+3x+6y)\right)=$$

$$=\frac{1}{3}\cdot\left(4+3x+6y+\frac{4}{3}\cdot 3+\frac{5}{3}\cdot 6\right)=\frac{1}{3}\cdot(4+3x+6y+4+10)=$$

$$=\frac{1}{3}\cdot(18+3x+6y)=6+x+2y$$

Then,

$$\boxed{P.I.=6+x+2y}$$

Conclusion,

$$z(x,y)=C.F.+P.I.=e^{x}\cdot\varphi_1(y-x)+e^{3x}\cdot\varphi_2(y-2x)+6+x+2y$$

$$\boxed{\begin{cases} z(x,y)=e^{x}\cdot\varphi_1(y-x)+e^{3x}\cdot\varphi_2(y-2x)+6+x+2y \\ where\ \varphi_1 and\ \varphi_2\ are\ arbitrary\ functions \end{cases}}$$

ANSWER

1)

$$(D^2+D-1)z=4e^{x+y}\cos(x+y)\rightarrow$$

$$\begin{cases} z(x,y)=e^{-\frac{1+\sqrt{5}}{2}\cdot x}\cdot\varphi_1(y)+e^{-\frac{1-\sqrt{5}}{2}x}\cdot\varphi_2(y)-\frac{4}{5}e^{x+y}(3\sin(x+y)+2\cos(x+y)) \\ where\ \varphi_1 and\ \varphi_2\ are\ arbitrary\ functions \end{cases}$$

2)

$$(D+D'-1)(D+2D'-3)z=4+3x+6y\rightarrow$$

$$\begin{cases} z(x,y)=e^{x}\cdot\varphi_1(y-x)+e^{3x}\cdot\varphi_2(y-2x)+6+x+2y \\ where\ \varphi_1 and\ \varphi_2\ are\ arbitrary\ functions \end{cases}$$