Answer on Question #79911 – Math – Linear Algebra

Question

Find inverse of the matrix B in part a) of the question by finding the adjoint as well as using Cayley-Hamilton theorem.

$$B = \begin{bmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution

0. $\det B = \begin{vmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & -3 \\ 2 & 4 \end{vmatrix} = 2 \cdot (-4+6) = 4 \neq 0$

1. $B^{-1} = \frac{1}{\det B} \operatorname{adj} B = \frac{1}{2\cdot(-4+6)} \begin{bmatrix} 8 & * & * \\ * & -2 & * \\ * & * & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

Recall that the adjoint of a matrix is the transpose of its cofactor matrix:

$$\operatorname{adj} B = \left[\text{Cofactor of } b_{ij}\right]^T = \left[(-1)^{i+j}(\text{Minor of } b_{ij})\right]^T = \left[(-1)^{i+j}(\text{Minor of } b_{ji})\right]$$
$$= \begin{bmatrix} b_{22}b_{33} - b_{23}b_{32} & -(b_{12}b_{33} - b_{13}b_{32}) & -(b_{12}b_{23} - b_{13}b_{22}) \\ -(b_{21}b_{33} - b_{23}b_{31}) & b_{11}b_{33} - b_{13}b_{31} & -(b_{11}b_{23} - b_{13}b_{21}) \\ -(b_{21}b_{32} - b_{22}b_{31}) & -(b_{11}b_{32} - b_{12}b_{31}) & b_{11}b_{22} - b_{12}b_{21} \end{bmatrix}$$

2. $p(\lambda) = \det(\lambda I - B) = (-1)^3 \det(B - \lambda I) = -\det\begin{bmatrix} -1-\lambda & -3 & 0 \\ 2 & 4-\lambda & 0 \\ -1 & -1 & 2-\lambda \end{bmatrix} = -(2-\lambda)(6+\lambda^2-3\lambda-4) = (\lambda-2)(\lambda^2-3\lambda+2) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$

$p(B) = 0$ (Cayley-Hamilton theorem)

$4B^{-1} = B^2 - 5B + 8I$, where $8I = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$B^2 = \begin{bmatrix} -5 & -9 & 0 \\ 6 & 10 & 0 \\ -3 & -3 & 4 \end{bmatrix}$, $-5B + 8I = \begin{bmatrix} 13 & 15 & 0 \\ -10 & -12 & 0 \\ 5 & 5 & -2 \end{bmatrix}$, $4B^{-1} = \begin{bmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{4}\begin{bmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Answer:

The inverse is $B^{-1}=\frac{1}{2}\begin{bmatrix}4 & 3 & 0\\ -2 & -1 & 0\\ 1 & 1 & 1\end{bmatrix}$.