Answer on Question #79909 – Math – Linear Algebra

Question

Let P_3 be the inner product space of polynomials of degree at most 3 over $\mathbb{R}$ with respect to the inner product $\int_0^1 f(x)g(x)\,dx$. Apply the Gram-Schmidt orthogonalisation process to find an orthonormal basis for the subspace of P_3 generated by the vectors $\{1-2x, 2x+6x^2, -3x^2+4x^3\}$

Solution

1. Obviously these vectors are linearly independent.

2.

a. $u_1 = v_1 = \mathbf{1-2x}$

$\langle u_1, u_1\rangle = \int_0^1 (1-4x+4x^2)\,dx = 4/3 - 4/2 - 1 = \mathbf{1/3}$

b.

$\langle u_1, v_2\rangle = \int_0^1 (1-2x)(2x+6x^2)\,dx = \int_0^1(-12x^3+2x^2+2x)\,dx = -12/4 + 2/3 + 1 = -4/3$

$\langle u_2, u_2\rangle = \int_0^1(36x^4 - 72x^3 + 84x^2 - 48x + 16)\,dx = 36/5 - 18 + 28 - 24 + 16 = \mathbf{46/5}$

$$u_2 = v_2 - \frac{\langle u_1,v_2\rangle}{\langle u_1,u_1\rangle}u_1 = 2x + 6x^2 + 4 - 8x = \mathbf{4 - 6x + 6x^2}$$

c.

$\langle u_1, v_3\rangle = \int_0^1(1-2x)(-3x^2+4x^3)\,dx = \int_0^1(-8x^4+10x^3-3x^2)\,dx = -8/5 + 10/4 - 1 = -1/10$

$\langle u_2, v_3\rangle = \int_0^1(4-6x+6x^2)(-3x^2+4x^3)\,dx = \int_0^1(24x^5 - 42x^4 + 34x^3 - 12x^2)\,dx = 4 - 42/5 + 34/4 - 4 = -2/5 + 2/4 = 1/10$

$\langle u_3, u_3\rangle = \int_0^1(16x^6 - 564/23\,x^5 + 54141/10580\,x^4 + 28199/5290\,x^3 - 68061/52900\,x^2 - 7257/26450\,x + 3481/52900)\,dx = 16/7 - 94/23 + 54141/52900 + 140995/(2*52900) - 22687/52900 - 7257/52900 + 3481/52900 = -290/161 + 196351/2*52900 = \mathbf{1759/32200}$

$$u_3 = v_3 - \frac{\langle u_1,v_3\rangle}{\langle u_1,u_1\rangle}u_1 - \frac{\langle u_2,v_3\rangle}{\langle u_2,u_2\rangle}u_2 = -3x^2 + 4x^3 + 3/10 - 6/10\,x - 4/92 + 6/92\,x - 6/92\,x^2 = \mathbf{59/230 - 123/230\,x - 141/46\,x^2 + 4x^3}$$

Answer:

$b_1 = \sqrt{3}(1-2x)$, $b_2 = \sqrt{5/46}\,(4-6x+6x^2)$,

$b_3 = 80\sqrt{5/1759}\,(59/230 - 123/230\,x - 141/46\,x^2 + 4x^3)$.