Answer on Question #79881 – Math – Calculus

Question

Trace the curve:

$$x^3 + y^3 = 3ax^2.$$

Solution

If $a = 0$

$$x^3 + y^3 = 0$$
$$(x + y)(x^2 - xy + y^2) = 0$$
$$y = -x, x \in R$$

The function $y = -x$, which is just a straight line with a slope of -1 and y −intercept at the origin.

If $a \neq 0$

1. Domain: $(-\infty, \infty)$

2. Symmetry

The curve is not symmetrical about the y −axis.
The curve is not symmetrical about the x −axis.
The curve is not symmetrical in opposite quadrants.
The curve is not symmetrical about the line $y = x$.

3. Origin.

The curve passes through the origin

$$x = 0 => y = 0$$

The equations of the tangents to the curve at the origin are obtained by equating the lowest degree terms in x and y in the given equation to zero.

Lowest degree term

$$3ax^2 = 0, a \neq 0$$
$$x = 0$$

The tangent: $x = 0$.
The tangents are real and distinct.
The origin is cusp of the first kind (simple cusp).

4. Intersection with the coordinate axes.

$y - intercept: x = 0 => y = 0, point(0, 0)$

$x - intercept: y = 0 => x^3 + 0 = 3ax^2, a \neq 0$

$x^2(x - 3a) = 0$

$x = 0 \quad or \quad x = 3a$

$point(0, 0), point(3a, 0)$

5. First derivative

Take derivative with respect to x of both sides of the equation and use the Chain rule

$$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(3ax^2)$$
$$3x^2+3y^2\frac{dy}{dx}=6ax$$
$$\frac{dy}{dx}=\frac{2ax-x^2}{y^2}$$
We have the vertical tangent $x=0$ at the $point(0,0)$.
We have the vertical tangent $x=3a$ at the $point(3a,0)$.

6. Asymptote(s)
$$x^3+y^3=3ax^2$$
$$y=\sqrt[3]{3ax^2-x^3}$$
There is no vertical asymptote.
There is no horizontal asymptote.

$$m=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}\frac{\sqrt[3]{3ax^2-x^3}}{x}=\lim_{x\to\pm\infty}\sqrt[3]{\frac{3ax^2}{x^3}-\frac{x^3}{x^3}}=\lim_{x\to\pm\infty}\sqrt[3]{\frac{3a}{x}-1}=$$
$$=\sqrt[3]{0-1}=-1$$
$$b=\lim_{x\to\pm\infty}\left(y-(-x)\right)=\lim_{x\to\pm\infty}\left(\sqrt[3]{3ax^2-x^3}+x\right)=$$
$$=\lim_{x\to\pm\infty}\left(\left(\sqrt[3]{3ax^2-x^3}+x\right)\left(\frac{\left(\sqrt[3]{3ax^2-x^3}\right)^2-x\sqrt[3]{3ax^2-x^3}+x^2}{\left(\sqrt[3]{3ax^2-x^3}\right)^2-x\sqrt[3]{3ax^2-x^3}+x^2}\right)\right)=$$
$$=\lim_{x\to\pm\infty}\left(\frac{3ax^2-x^3+x^3}{\left(\sqrt[3]{3ax^2-x^3}\right)^2-x\sqrt[3]{3ax^2-x^3}+x^2}\right)=$$
$$=\lim_{x\to\pm\infty}\left(\frac{\frac{3ax^2}{x^2}}{\left(\sqrt[3]{\frac{3ax^2}{x^3}-\frac{x^3}{x^3}}\right)^2-\frac{x}{x}\sqrt[3]{\frac{3ax^2}{x^3}-\frac{x^3}{x^3}}+\frac{x^2}{x^2}}\right)=$$
$$=\lim_{x\to\pm\infty}\left(\frac{3a}{\left(\sqrt[3]{\frac{3a}{x}-1}\right)^2-\sqrt[3]{\frac{3a}{x}-1}+1}\right)=\frac{3a}{\left(\sqrt[3]{0-1}\right)^2-\sqrt[3]{0-1}+1}=a$$

There is slant (oblique) asymptote
$$y=-x+a$$

7. Increasing and decreasing

$$\frac{dy}{dx} = \frac{2ax - x^2}{y^2}, -1 < x \leq 3$$

$$\frac{dy}{dx} = 0 => \frac{2ax - x^2}{y^2} = 0$$

$$2ax - x^2 = 0, y \neq 0$$

$$If\ x = 0, y = 0,\ \ \frac{dy}{dx}\ is\ undefined$$

We take $x = 2a$

$$y|_{x=2a} = \sqrt[3]{3a(2a)^2 - (2a)^3} = \sqrt[3]{4}a$$

$a > 0$

$$If\ x < 0,\ \ \frac{dy}{dx} < 0,\ \ y\ decreases$$

$$If\ 0 < x < 2a,\ \ \frac{dy}{dx} > 0,\ \ y\ increases$$

$$If\ x > 2a,\ \ \frac{dy}{dx} < 0,\ \ y\ decreases$$

The function y has the local maximum with value of $\sqrt[3]{4}a$ at $x = 2a$.
The function y has the local minimum with value of 0 at $x = 0$.

$a < 0$

$$If\ x < 2a,\ \ \frac{dy}{dx} < 0,\ \ y\ decreases$$

$$If\ 2a < x < 0,\ \ \frac{dy}{dx} > 0,\ \ y\ increases$$

$$If\ x > 0,\ \ \frac{dy}{dx} < 0,\ \ y\ decreases$$

The function y has the local maximum with value of 0 at $x = 0$.
The function y has the local minimum with value of $\sqrt[3]{4}a$ at $x = 2a$.

8. Sketch the graph