Answer on Question #79871 – Math – Linear Algebra

Question

Let P3 be the inner product space of polynomials of degree at most 3 over R with respect to the inner product
hf,gi =
Z 1
0
f(x)g(x)dx.
Apply the Gram-Schmidt orthogonalisation process to find an orthonormal basis for the subspace of P3 generated by the vectors (8)

1−2x,2x+6x
2
,−3x
2 +4x
3

Solution

$a_1 = 1 - 2x, a_2 = 2x + 6x^2, a_3 = -3x^2 + 4x^3.$

Apply the Gram-Schmidt orthogonalisation process:

$b_1 = a_1 = 1 - 2x,$

$$b_2 = a_2 - \frac{\langle a_2, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 .$$

$$\langle a_2, b_1 \rangle = \int_0^1 \left(2x + 6x^2\right)\left(1 - 2x\right) dx = \int_0^1 \left(2x + 2x^2 - 12x^3\right) dx =$$

$$= 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} - 12 \cdot \frac{1}{4} = -\frac{4}{3}$$

$$\langle b_1, b_1 \rangle = \int_0^1 \left(1 - 2x\right)^2 dx = \int_0^1 \left(1 - 4x + 4x^2\right) dx =$$

$$= 1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} = \frac{1}{3}$$

$$b_2 = 2x + 6x^2 - \frac{-4/3}{1/3}\left(1 - 2x\right) = 4 - 6x + 6x^2$$

$$b_3 = a_3 - \frac{\langle a_3, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 - \frac{\langle a_3, b_2 \rangle}{\langle b_2, b_2 \rangle} b_2$$

$$\langle a_3, b_1 \rangle = \int_0^1 \left(-3x^2 + 4x^3\right)\left(1 - 2x\right) dx = \int_0^1 \left(-3x^2 + 10x^3 - 8x^4\right) dx =$$

$$= -3 \cdot \frac{1}{3} + 10 \cdot \frac{1}{4} - 8 \cdot \frac{1}{5} = -\frac{1}{10}$$

$$\langle b_2, b_2\rangle = \int_0^1 \left(4-6x+6x^2\right)^2 dx = \int_0^1 \left(16-48x+84x^2-72x^3+36x^4\right)dx =$$

$$= 16-48\cdot\frac{1}{2}+84\cdot\frac{1}{3}-72\cdot\frac{1}{4}+36\cdot\frac{1}{5}=\frac{46}{5}$$

$$b_3 = -3x^2+4x^3-\frac{-1/10}{1/3}(1-2x)-\frac{1/10}{46/5}\left(4-6x+6x^2\right)=\frac{59}{230}-\frac{123}{230}x-\frac{141}{46}x^2+4x^3.$$

We need to normalize all these vectors:

$$\langle b_3, b_3\rangle = \frac{1}{230^2}\int_0^1 (59-123x-705x^2+920x^3)^2 dx =$$

$$= \frac{1}{230^2}\int_0^1 (59^2+123^2x^2+705^2x^4+920^2x^6-2\cdot 59\cdot 123x-2\cdot 59\cdot 705x^2+2\cdot 59\cdot 920x^3+$$

$$+2\cdot 123\cdot 705x^3-2\cdot 123\cdot 920x^4-2\cdot 705\cdot 920x^5)dx = \frac{40457}{14\cdot 230^2}$$

$$e_1 = \frac{b_1}{\sqrt{\langle b_1, b_1\rangle}} = \frac{1-2x}{\sqrt{1/3}} = \sqrt{3}-2\sqrt{3}x$$

$$e_2 = \frac{b_2}{\sqrt{\langle b_2, b_2\rangle}} = \frac{4-6x+6x^2}{\sqrt{46/5}} = 2\sqrt{\frac{5}{46}}\left(2-3x+3x^2\right)$$

$$e_3 = \frac{b_3}{\sqrt{\langle b_3, b_3\rangle}} = \frac{59-123x-705x^2+920x^3}{\sqrt{40457/14}} = \sqrt{\frac{14}{40457}}\left(59-123x-705x^2+920x^3\right).$$

Answer:

$$e_1 = \frac{b_1}{\sqrt{\langle b_1, b_1\rangle}} = \frac{1-2x}{\sqrt{1/3}} = \sqrt{3}-2\sqrt{3}x$$

$$e_2 = \frac{b_2}{\sqrt{\langle b_2, b_2\rangle}} = \frac{4-6x+6x^2}{\sqrt{46/5}} = 2\sqrt{\frac{5}{46}}\left(2-3x+3x^2\right)$$

$$e_3 = \frac{b_3}{\sqrt{\langle b_3, b_3\rangle}} = \frac{59-123x-705x^2+920x^3}{\sqrt{40457/14}} = \sqrt{\frac{14}{40457}}\left(59-123x-705x^2+920x^3\right).$$