Answer on Question #79870 – Math – Linear Algebra

Question

Find the orthogonal canonical reduction of the quadratic form

$$Q = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$

Also, find its principal axes.

Solution

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$-\det(\lambda I - A) = \det(A - \lambda I) = \left|\begin{bmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{bmatrix}\right| = (1-\lambda)(-2\lambda + \lambda^2) - (-1)(-2 + \lambda) + (-1)(2-\lambda) = (\lambda - 2)(-\lambda^2 + \lambda + 1 + 1) = (\lambda - 2)(2-\lambda)(\lambda + 1) = 0$$

$\lambda_1 = \lambda_2 = 2,\ \lambda_3 = -1$ are eigenvalues.

$Q = 2a^2 + 2b^2 - c^2$ is the orthogonal canonical reduction.

$$\operatorname{Ker}\begin{bmatrix} 1-2 & -1 & -1 \\ -1 & 1-2 & -1 \\ -1 & -1 & 1-2 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\} =$$

$$= \operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}.$$

$$\operatorname{Ker}\begin{bmatrix} 1+1 & -1 & -1 \\ -1 & 1+1 & -1 \\ -1 & -1 & 1+1 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} 2 & -1 & -1 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} =$$

$$\operatorname{Ker}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\};$$

$p_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, p_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $p_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are the principal axes.

Answer:

1. $Q = 2a^2 + 2b^2 - c^2$ is the orthogonal canonical reduction.
2. $p_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, p_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $p_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are the principal axes.