Answer on Question #79868 – Math – Linear Algebra

Question

Consider the basis $e_1 = (-2,4,-1), e_2 = (-1,3,-1)$ and $e_3 = (1,-2,1)$ of $\mathbb{R}^3$ over $\mathbb{R}$. Find the dual basis of $\{e_1, e_2, e_3\}$.

Solution

For every basis $\{e_1, e_2, e_3\}$ there is a basis $\{v_1, v_2, v_3\}$ such that

$$v_i(e_j) = \begin{cases} 1 & if \ \ i \neq j \\ 0 & if \ \ i = j \end{cases}$$

The basis $\{v_1, v_2, v_3\}$ is known as the dual basis of $\{e_1, e_2, e_3\}$. Clearly, it is unique.

$$\langle v_i, e_j \rangle = \delta_{ij}$$

$$AM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

$$A = M^{-1}$$

$$\begin{pmatrix} -2 & 4 & -1 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1/(-2)} \begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+(2)R_2} \begin{pmatrix} 1 & 0 & -1/2 & -3/2 & 2 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1/2 & -3/2 & 2 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+R_3} \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+R_3} \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{(2)R_3} \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

$$A = M^{-1} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$v_1 = (-1, 0, 1)$,
$v_2 = (2, 1, 0)$,
$v_3 = (1, 1, 2)$.