Question

Write abstract of Runge Kutta Method and Runge Kutta Fehlberg Method.

Solution

The Runge-Kutta methods are a family of iterative methods, which include the well-known routine called the Euler Method, used in temporal discretization for the approximate solutions of ordinary differential equations. These methods were developed around 1900 by the German mathematicians C. Runge and M. W. Kutta.

The classical Runge-Kutta method

The classical Runge-Kutta method (or simply the Runge–Kutta method) is most widely known member of the Runge-Kutta family. It is generally referred to RK4 [*Wolfram MathWorld*, *Wikipedia*].

Let an initial value problem be specified as follows:

$$\frac{dy}{dx} = f(x, y), \ y(x_0) = y_0. \tag{1}$$

Then, for a chosen step h the approximate solution of the problem is given by:

$$x_{n+1} = x_n + h,$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$

$$k_4 = hf(x_n + h, y_n + k_3).$$

The RK4 method is a fourth-order method, meaning that the local truncation error is on the order of $O(h^5)$, while the total accumulated error is on the order of $O(h^4)$.

The Runge-Kutta-Fehlberg method

The Runge-Kutta-Fehlberg method was developed by the German mathematician Erwin Fehlberg and is based on the classical Runge–Kutta method. It is referred to RKF45. The idea of the Runge-Kutta-Fehlberg method is to solve the problem twice using step sizes h and $h/2$ and compare answers at the mesh points corresponding to the larger step size [*Mathew&Fink*, *Wikipedia*,].

To approximate the solution of the problem (1), the following algorithm should be implemented:

$$x_{n+1} = x_n + h,$$

$$y_{n+1}^{[4]} = y_n + \left(\frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5\right), \qquad error\ of\ O(h^4),$$

$$y_{n+1} = y_n + \left(\frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6\right), \qquad error\ of\ O(h^5),$$

$$\varepsilon = \left|y_{n+1}^{[5]} - y_{n+1}^{[4]}\right|,$$

where

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf\left(x_n + \frac{1}{4}h, y_n + \frac{1}{4}k_1\right),$$

$$k_3 = hf\left(x_n + \frac{3}{8}h, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2\right),$$

$$k_4 = hf\left(x_n + \frac{12}{13}h, y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right),$$

$$k_5 = hf\left(x_n + h, y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right),$$

$$k_6 = hf\left(x_n + \frac{1}{2}h, y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right).$$

Value ε gives an error estimate for y_{n+1} at x_{n+1}. If the error estimate exceeds the goal tolerance, the step h should be decreased, and y_{n+1} should be recalculated at new x_{n+1}.

References:

1. Wolfram MathWorld. *Runge-Kutta Method*. [online] Available at: http://mathworld.wolfram.com/Runge-KuttaMethod.html [Accessed 15 Aug 2018].
2. Wikipedia. *Runge–Kutta methods*. [online] Available at: https://en.wikipedia.org/wiki/Runge-Kutta_methods [Accessed 15 Aug 2018].
3. Wikipedia. Runge–Kutta–Fehlberg method. [online] Available at: https://en.wikipedia.org/wiki/Runge-Kutta-Fehlberg_method [Accessed 15 Aug 2018].
4. John H. Mathews and Kurtis K. Fink. *Numerical Methods Using Matlab*, 4th Edition, 2004.