Answer on Question #79776 – Math – Differential Equations

Question

Solve the equation

$$x\frac{dy}{dx} - ay = x + 1,$$

where a is a constant.

$$a)\quad y = \frac{x}{1-a} - \frac{1}{a} + Cx;$$

$$b)\ y = \frac{x}{1-a} + \frac{1}{a} + Cx;$$

$$c)\ y = \frac{x}{1+a} - \frac{1}{a} + Cx;$$

$$d)\ y = -\frac{x}{1-a} - \frac{1}{a} + Cx\,.$$

Solution

Let's write the equation in the form:

$$\frac{dy}{dx} - \frac{a}{x}y = 1 + \frac{1}{x}.$$

It's a first order linear differential equation, in turn that means it looks like

$$\frac{dy}{dx} + P(x)y = Q(x)$$

with $P(x) = -\frac{a}{x}$ and $Q(x) = 1 + \frac{1}{x}$.

It can be integrated in three ways.

1) The first way is using auxiliary function.

Let

$$y = uv,$$

where $u = u(x)$ and $v = v(x)$ are the auxiliary functions.

Then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

So we have:

$$\frac{du}{dx}v + u\frac{dv}{dx} - \frac{a}{x}uv = 1 + \frac{1}{x}$$

or

$$\frac{du}{dx}v + u\left(\frac{dv}{dx} - \frac{a}{x}v\right) = 1 + \frac{1}{x}.$$

Since one of the auxiliary functions can be chosen arbitrarily, let's choose a function v such that it will satisfy the condition:

$$\frac{dv}{dx} - \frac{a}{x}v = 0.$$

So we have

$$\frac{dv}{dx} = \frac{a}{x}v\,.$$

It's a separable equation what can be easily integrated:

$$\int \frac{dv}{v} = a\int \frac{dx}{x};$$

$$ln|v| = aln|x|$$

$$v = x^{a}\,.$$

Substitute the expression for the function v in equation:

$$\frac{du}{dx}x^{a} = 1 + \frac{1}{x}$$

or

$$\frac{du}{dx} = \frac{1}{x^{a}} + \frac{1}{x^{a+1}}.$$

It's a separable equation so we can integrate it:

$$\int du = \int \left(\frac{1}{x^{a}} + \frac{1}{x^{a+1}}\right)dx;$$

$$u = \frac{x^{1-a}}{1-a} - \frac{x^{-a}}{a} + C = \frac{x}{(1-a)x^{a}} - \frac{1}{ax^{a}} + C, \qquad where\ C = const.$$

After reverse replacement we have a general solution of the given equation:

$$y = \frac{x}{1-a} - \frac{1}{a} + Cx^{a}, \qquad where\ C = const.$$

2) <u>The second way</u> is a use of an integrating factor

$$\mu(x) = e^{\int P(x)dx}.$$

The solution is then commonly written as

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx.$$

Determine the integrating factor:

$$\mu(x) = e^{-a\int\frac{dx}{x}} = e^{-aln|x|} = x^{-a}.$$

Then the solution is

$$y = x^a \int x^{-a}\left(1 + \frac{1}{x}\right) dx = x^a \int \left(\frac{1}{x^a} + \frac{1}{x^{a+1}}\right) dx = x^a \left(\frac{x^{1-a}}{1-a} - \frac{x^{-a}}{a} + C\right) =$$

$$= \frac{x}{1-a} - \frac{1}{a} + Cx^a.$$

So the general solution of the given equation is

$$y = \frac{x}{1-a} - \frac{1}{a} + Cx^a, \qquad where\ C = const.$$

3) <u>The third way</u> is a use of the variation of constants method.

We start with the homogeneous equation:

$$\frac{dy}{dx} - \frac{a}{x} y = 0,$$

which is a separable equation, so we can integrate it:

$$\int \frac{dy}{y} = a \int \frac{dx}{x};$$

$$ln|y| = aln|x| + lnC;$$

$$y = Cx^a.$$

The idea of the variation of constants method is to replace constant C by a function $C(x)$.

So we have:

$$y = C(x)x^a$$

and

$$\frac{dy}{dx} = \frac{dC}{dx} x^a + Cax^{a-1}.$$

Substitute the expression for y and $\frac{dy}{dx}$ in the equation:

$$x\left(\frac{dC}{dx} x^a + Cax^{a-1}\right) - aCx^a = x + 1;$$

$$\frac{dC}{dx} x^{a+1} + Cax^a - aCx^a = x + 1;$$

$$\frac{dC}{dx} x^{a+1} = x + 1.$$

The obtained equation is a separable equation and can be easily integrated:

$$C(x) = \int \left(\frac{1}{x^a} + \frac{1}{x^{a+1}}\right) dx = \frac{x^{1-a}}{1-a} - \frac{x^{-a}}{a} + C_1,$$

and the general solution of the given equation is:

$$y = \left(\frac{x^{1-a}}{1-a} - \frac{x^{-a}}{a} + C_1\right) x^a = \frac{x}{1-a} - \frac{1}{a} + C_1 x^a.$$

The result is evidently the same as what we found using a substitution (an auxiliary function) or an integrating factor.