ANSWER on Question #79736 – Math – Calculus

QUESTION

Evaluate the integral

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx$$

SOLUTION

To solve this problem, we recall the definition of the beta function

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

Or

$$B(x,y)=\int_0^\infty \frac{t^{x-1}}{(t+1)^{x+y}}dt$$

Let us try to bring the given integral to this form. We introduce a substitution

$$t=\frac{1-x^4}{1+x^4}$$

Let us see how to change the integrand and the boundaries of integration.

1. Integration limits.

$$x=0\to t=\frac{1-0^4}{1+0^4}=1\to \boxed{x=0\to t=1}$$

$$x=1\to t=\frac{1-1^4}{1+1^4}=\frac{1-1}{1+1}=\frac{0}{2}=0\to \boxed{x=1\to t=0}$$

2. Integrand.

$$t=\frac{1-x^4}{1+x^4}\to t(1+x^4)=1-x^4\to t+tx^4=1-x^4\to tx^4+x^4=1-t\to$$

$$x^4(t+1)=1-t\to \boxed{x^4=\frac{1-t}{1+t}\to x=\left(\frac{1-t}{1+4}\right)^{\frac{1}{4}}}$$

$$x=\left(\frac{1-t}{1+4}\right)^{\frac{1}{4}}\to dx=\frac{1}{4}\cdot\left(\frac{1-t}{1+t}\right)^{\frac{1}{4}-1}\cdot\frac{-1\cdot(t+1)-1\cdot(1-t)}{(1+t)^2}dt\to$$

$$dx=\frac{1}{4}\cdot\left(\frac{1-t}{1+t}\right)^{-\frac{3}{4}}\cdot\frac{-t-1-1+t}{(1+t)^2}dt\to dx=\frac{1}{4}\cdot\left(\frac{1-t}{1+t}\right)^{-\frac{3}{4}}\cdot\frac{-2}{(1+t)^2}dt\to$$

$$\boxed{dx = -\frac{1}{2}\cdot\left(\frac{1-t}{1+t}\right)^{-\frac{3}{4}}\cdot\frac{dt}{(1+t)^2}}$$

$$1-x^4 = 1-\frac{1-t}{1+t} = \frac{1+t-(1-t)}{1+t} = \frac{1+t-1+t}{1+t} = \frac{2t}{1+t} \rightarrow \boxed{1-x^4 = \frac{2t}{1+t}}$$

$$1+x^4 = 1+\frac{1-t}{1+t} = \frac{1+t+(1-t)}{1+t} = \frac{1+t+1-t}{1+t} = \frac{2}{1+t} \rightarrow \boxed{1+x^4 = \frac{2}{1+t}}$$

Then

$$\frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = \frac{\left(\frac{2t}{1+t}\right)^{\frac{3}{4}}}{\left(\frac{2}{1+t}\right)^2}\cdot\left(-\frac{1}{2}\right)\cdot\left(\frac{1-t}{1+t}\right)^{-\frac{3}{4}}\cdot\frac{dt}{(1+t)^2} \rightarrow$$

$$\frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = -\frac{1}{2}\cdot\frac{(2t)^{\frac{3}{4}}\cdot(1+t)^2}{2^2\cdot(1+t)^{\frac{3}{4}}}\cdot\frac{(1-t)^{-\frac{3}{4}}}{(1+t)^{-\frac{3}{4}}}\cdot\frac{dt}{(1+t)^2} \rightarrow$$

$$\frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = -\frac{1}{2^1}\cdot\frac{2^{\frac{3}{4}}\cdot(t)^{\frac{3}{4}}}{2^2}\cdot\frac{(1-t)^{-\frac{3}{4}}}{\underbrace{(1+t)^{\frac{3}{4}-\frac{3}{4}}}_{=(1+t)^0\equiv 1}}\cdot\underbrace{\frac{(1+t)^2}{(1+t)^2}}_{\equiv 1}dt \rightarrow$$

$$\frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = -2^{\frac{3}{4}-1-2}\cdot t^{\frac{3}{4}}\cdot(1-t)^{-\frac{3}{4}}\cdot dt = -2^{\frac{3}{4}-1-2}\cdot t^{\frac{7}{4}-1}\cdot(1-t)^{\frac{1}{4}-1}\cdot dt \rightarrow$$

$$\frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = -2^{\frac{3-4-8}{4}}\cdot t^{\frac{7}{4}-1}\cdot(1-t)^{\frac{1}{4}-1}\cdot dt \rightarrow$$

$$\boxed{\frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = -2^{\frac{-9}{4}}\cdot t^{\frac{7}{4}-1}\cdot(1-t)^{\frac{1}{4}-1}\cdot dt}$$

Then

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = \int_1^0 -2^{\frac{-9}{4}}\cdot t^{\frac{7}{4}-1}\cdot(1-t)^{\frac{1}{4}-1}\cdot dt \rightarrow$$

$$\left[\begin{matrix}\textit{We use the next property of a definite integral}\\ \int_a^b f(x)dx = -\int_b^a f(x)dx\end{matrix}\right]$$

Then

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = -\left(-2^{-\frac{9}{4}}\right)\int_0^1 t^{\frac{7}{4}-1}\cdot(1-t)^{\frac{1}{4}-1}\cdot dt \equiv 2^{-\frac{9}{4}}\cdot B\left(\frac{7}{4},\frac{1}{4}\right)$$

Conclusion,

$$\boxed{\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = 2^{-\frac{9}{4}}\cdot B\left(\frac{7}{4},\frac{1}{4}\right)}$$

ANSWER:

$$\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2}dx = 2^{-\frac{9}{4}}\cdot B\left(\frac{7}{4},\frac{1}{4}\right)$$