ANSWER on Question #79724 – Math – Differential Equations

QUESTION

differential equations in the form of $F(x, y, p)$, $F(y/x, p)$ or differential equations containing $p(p = dy/dx)$

$$\mathbf{1.}\,(y - xp)^2 = b^2 + a^2p^2$$

$$\mathbf{2.}\,y^2p^2 + 4y^2 - (x + yp)^2 = 0$$

SOLUTION

1. Solve equation $y^2p^2 + 4y^2 - (x + yp)^2 = 0$

$$y^2p^2 + 4y^2 - (x + yp)^2 = 0 \to y^2p^2 + 4y^2 - x^2 - 2xyp - y^2p^2 = 0 \to 2xyp = 4y^2 - x^2 \to$$

$$2xyp = 4y^2 - x^2 | \div (x) \to 2yp = \frac{4y^2}{x} - x \to \boxed{2y \cdot \frac{dy}{dx} = \frac{4y^2}{x} - x \textit{ is Bernoulli differential equation}}$$

(More information: https://en.wikipedia.org/wiki/Bernoulli_differential_equation)

In our case,

$$2y \cdot \frac{dy}{dx} = \frac{4y^2}{x} - x \to \begin{bmatrix} \textit{We introduce the substitution} \\ y^2 = u \to 2y \cdot \frac{dy}{dx} = \frac{du}{dx} \end{bmatrix} \to \boxed{\frac{du}{dx} = \frac{4u}{x} - x \textit{ is inhomogeneous equation}}$$

1 STEP: Solve the homogeneous equation

$$\frac{du}{dx} = \frac{4u}{x} \to \frac{du}{u} = \frac{4dx}{x} \to \ln|u| = 4\ln|x| + \ln|C| \to u = C \cdot x^4$$

2 STEP: Solve the inhomogeneous equation.

We apply the method of variation of the constant.

$$C = C(x) \to u = C(x) \cdot x^4 \to \frac{du}{dx} = \frac{dC}{dx} \cdot x^4 + 4Cx^3$$

Then,

$$\frac{du}{dx} = \frac{4u}{x} - x \to \frac{dC}{dx} \cdot x^4 + 4Cx^3 = \frac{4C \cdot x^4}{x} - x \to \frac{dC}{dx} \cdot x^4 + 4Cx^3 = 4Cx^3 - x \to$$

$$\frac{dC}{dx} \cdot x^4 = -x \to dC = -\frac{dx}{x^3} \to \boxed{C(x) = \frac{1}{2x^2} + C}$$

Conclusion,

$$u = C(x) \cdot x^4 = x^4 \cdot \left(\frac{1}{2x^2} + C\right) \to \boxed{u = \frac{x^2}{2} + C \cdot x^4}$$

Then,

$$\begin{cases} u = \dfrac{x^2}{2} + C \cdot x^4 \\ u = y^2 \end{cases} \to y^2 = \frac{x^2}{2} + C \cdot x^4 \to \boxed{y = \pm\sqrt{\frac{x^2}{2} + C \cdot x^4}}$$

2. Solve equation $(y - xp)^2 = b^2 + a^2p^2$

1 STEP: Let's look at this equation as an algebraic one. We express the variable y in terms of x, p, a, b.

$$(y - xp)^2 = b^2 + a^2p^2 \to y - xp = \pm\sqrt{b^2 + a^2p^2} \to \boxed{y = xp \pm \sqrt{b^2 + a^2p^2}}$$

2 STEP: We differentiate the resulting equation, remembering that $p = {dy}/{dx}$

$$\left(Note: p' = \frac{dp}{dx}\right)$$

$$\frac{d}{dx} \times \Big| y = xp \pm \sqrt{b^2 + a^2p^2} \to \frac{dy}{dx} = p = \frac{d}{dx}\left(xp \pm \sqrt{b^2 + a^2p^2}\right) = p + xp' \pm \frac{2a^2p \cdot p'}{2\sqrt{b^2 + a^2p^2}} \to$$

$$p = p + xp' \pm \frac{a^2p \cdot p'}{\sqrt{b^2 + a^2p^2}} \to xp' \pm \frac{a^2p \cdot p'}{\sqrt{b^2 + a^2p^2}} = 0 \to p' \cdot \left(x \pm \frac{a^2p}{\sqrt{b^2 + a^2p^2}}\right) = 0 \to$$

$$\boxed{\left[\begin{array}{l} 1\ case:\ \ p' = 0 \\ 2case:\ \ x \pm \dfrac{a^2p}{\sqrt{b^2 + a^2p^2}} = 0 \end{array}\right.}$$

1 case: $p' = 0$

$$p' = 0 \to p = Const = C_1 \to \left(p = \frac{dy}{dx}\right) \to \frac{dy}{dx} = C_1 \to dy = C_1 dx \to \boxed{y = C_1x + C_2}$$

It remains to determine the constants C_1 and C_2. We substitute the solution found in the initial equation:

$$\begin{cases} (y - xp)^2 = b^2 + a^2p^2 \\ p = C_1 \\ y = C_1x + C_2 \end{cases} \to (C_1x + C_2 - C_1x)^2 = b^2 + a^2C_1^2 \to C_2^2 = b^2 + a^2C_1^2 \to$$

$$\boxed{\begin{cases} C_1 = c \\ C_2 = \pm\sqrt{b^2 + a^2c^2} \end{cases} \to y = cx \pm \sqrt{b^2 + a^2c^2}}$$

2 case:

$$x \pm \frac{a^2p}{\sqrt{b^2 + a^2p^2}} = 0 \to x = \mp \frac{a^2p}{\sqrt{b^2 + a^2p^2}}\Bigg| \times \sqrt{b^2 + a^2p^2} \to x\sqrt{b^2 + a^2p^2} = \mp a^2p \to$$

$$\left(x\sqrt{b^2 + a^2p^2}\right)^2 = (\mp a^2p)^2 \to x^2(b^2 + a^2p^2) = a^4p^2 \to x^2b^2 + a^2x^2p^2 = a^4p^2 \to$$

$$x^2b^2 = a^4p^2 - a^2x^2p^2 \to x^2b^2 = a^2(a^2 - x^2)p^2 | \div a^2(a^2 - x^2) \to p^2 = \frac{x^2b^2}{a^2(a^2 - x^2)} \to$$

$$\sqrt{p^2} = \sqrt{\frac{x^2b^2}{a^2(a^2 - x^2)}} \to p = \pm\frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}} \to \left(p = \frac{dy}{dx}\right) \to \frac{dy}{dx} = \pm\frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}} \to$$

$$dy = \pm\frac{b}{a} \cdot \frac{x \cdot dx}{\sqrt{a^2 - x^2}} \to y = \pm\frac{b}{a} \cdot \int \frac{xdx}{\sqrt{a^2 - x^2}} = \left[\begin{array}{c} a^2 - x^2 = t \\ -2xdx = dt \to xdx = -\frac{dt}{2} \end{array}\right] \to$$

$$y = \pm\frac{b}{a} \cdot \int \frac{-\frac{dt}{2}}{\sqrt{t}} \to y = \mp\frac{b}{a} \cdot \int \frac{dt}{2\sqrt{t}} \to y = \mp\frac{b}{a} \cdot \sqrt{t} + Const \to \boxed{y = \mp\frac{b}{a} \cdot \sqrt{a^2 - x^2} + Const}$$

Conclusion,

$$\boxed{\left[\begin{array}{c} y = cx \pm \sqrt{b^2 + a^2c^2} \\ or \\ y = \mp\frac{b}{a} \cdot \sqrt{a^2 - x^2} + Const \end{array}\right.}$$

ANSWER:

1. $\left[\begin{array}{c} y = cx \pm \sqrt{b^2 + a^2c^2} \\ or \\ y = \mp\frac{b}{a} \cdot \sqrt{a^2 - x^2} + Const \end{array}\right.$

2. $y = \pm\sqrt{\frac{x^2}{2} + C \cdot x^4}$