Question 1. *Let $T \in B(X, X)$, where X is a complex Banach space, and T be idempotent (i. e. $T^2 = T$). Prove that if T is neither 0 nor I, then $\sigma(T) = \{0, 1\}$.*

Solution. By definition the spectrum of T (denoted by $\sigma(T)$) consists of all $\lambda \in \mathbb{C}$, such that $T - \lambda I$ is not invertible.

First of all prove that $\{0, 1\} \subset \sigma(T)$. If $\lambda = 0$, then $T - \lambda I = T$, so we need to show that T is not invertible. Suppose there is T^{-1}. Then we can multiply the equation $T^2 = T$ by T^{-1} and obtain $T = I$, which is not the case of our task. Now consider $\lambda = 1$ and prove that $T - \lambda I = T - I$ is not invertible. Suppose the contrary. Note that the equality $T^2 = T$ can be written as $T(T - I) = 0$. The last one, being multiplied by $(T - I)^{-1}$, implies $T = 0$, which is a contradiction.

Now take $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and prove that $\lambda \notin \sigma(T)$, i. e. $T - \lambda I$ is invertible. Try to find its inverse in the form $\mu T - \nu I$ for some $\mu, \nu \in \mathbb{C}$. Note that $T - \lambda I$ and $\mu T - \nu I$ commute, because T and I commute. So, it is sufficient to find μ and ν such that $(T - \lambda I)(\mu T - \nu I) = I$. We have

$$(T - \lambda I)(\mu T - \nu I) = \mu T^2 - \nu T - \lambda \mu T + \lambda \nu I = (\mu - \nu - \lambda \mu) T + \lambda \nu I,$$

because $T^2 = T$. We need $\mu - \nu - \lambda \mu = 0$ and $\lambda \nu = 1$, so $\nu = \frac{1}{\lambda}$ and $\mu = \frac{\nu}{1-\lambda} = \frac{1}{\lambda(1-\lambda)}$. Since $\lambda \notin \{0, 1\}$, μ and ν are defined. Thus, $\frac{1}{\lambda}(\frac{1}{1-\lambda} T - I)$ is inverse to $T - \lambda I$ and hence $T - \lambda I$ is invertible. $\square$