Question 1. *Let $T \in B(X,X)$ be invertible in $B(X,X)$, where X is a complex Banach space. Show that $\sigma(T^{-1}) = \{1/\lambda : \lambda \in \sigma(T)\}$.*

Solution. By the definition of the spectrum, we need to prove that $T - \lambda I$ is not invertible if and only if $T^{-1} - \frac{1}{\lambda} I$ is not invertible. Note that $\lambda = 0$ does not belong to $\sigma(T)$, because T is invertible. Thus, our task is to prove that $T - \lambda I$ is invertible iff $T^{-1} - \frac{1}{\lambda} I$ is invertible for all $\lambda \neq 0$.

Let T and $T - \lambda I$ be invertible, $\lambda \neq 0$. Prove that $T^{-1} - \frac{1}{\lambda} I$ is invertible. It is sufficient to show that $T^{-1} - \frac{1}{\lambda} I$ is invertible on the left and on the right. Indeed, note that

$$-\lambda T \left(T^{-1} - \frac{1}{\lambda} I \right) = T - \lambda I.$$

Since $T - \lambda I$ is invertible, this implies

$$-\lambda \left(T - \lambda I\right)^{-1} T \left(T^{-1} - \frac{1}{\lambda} I \right) = I.$$

Furthermore

$$\left(T^{-1} - \frac{1}{\lambda} I \right) (-\lambda T)(T - \lambda I)^{-1} = (T - \lambda I)(T - \lambda I)^{-1} = I.$$

Thus, $-\lambda (T - \lambda I)^{-1} T = -\lambda T (T - \lambda I)^{-1} = (T^{-1} - \frac{1}{\lambda} I)^{-1}$.

The converse implication: "if $T^{-1} - \frac{1}{\lambda} I$ is invertible, then $T - \lambda I$ is invertible" is proved similarly. □