Question 1. *Let λ be an eigenvalue of $T \in B(X, X)$, where X is a complex Banach space. If P is a polynomial, show that $P(\lambda)$ is an eigenvalue of $P(T)$.*

Solution. First of all prove the following fact: if λ is an eigenvalue of T and μ is an eigenvalue of S, then $a\lambda + b\mu$ is an eigenvalue of $aT + bS$ for all $a, b \in \mathbb{C}$. Indeed, for any $x \in X$ we have

$$(aT + bS)x = aTx + bSx = a\lambda x + b\mu x = (a\lambda + b\mu)x.$$

Furthermore, show that λ^n is an eigenvalue of T^n, if λ is an eigenvalue of T (here n is a nonnegative integer). Use the induction on n. The case $n = 0$ is trivial: $T^n = I$ and $\lambda^n = \lambda^0 = 1$, which is the only eigenvalue of I. Suppose $n \geq 1$ and λ^{n-1} is an eigenvalue of T^{n-1}. Prove that λ^n is an eigenvalue of T^n. Let x be an eigenvector of T^{n-1}, corresponding to λ^{n-1}. Then

$$T^n x = T(T^{n-1}x) = T(\lambda^{n-1}x) = \lambda^{n-1}Tx = \lambda^{n-1} \cdot \lambda x = \lambda^n x,$$

therefore x is an eigenvector of T^n, corresponding to λ^n.

Now consider an arbitrary polynomial $P(x) = a_0 + a_1 x + \dots a_n x^n$ and an eigenvalue λ of T. Prove that $P(\lambda) = a_0 + a_1\lambda + \dots + a_n\lambda_n$ is an eigenvalue of $P(T) = a_0 I + a_1 T + \dots + a_n T^n$. Indeed, as was shown above, λ^k is an eigenvalue of T^k for all $k = 0, \dots, n$. Then $\sum_{k=0}^n a_k\lambda^k$ is an eigenvalue of $\sum_{k=0}^n a_k T^k = P(T)$. □