ANSWER on Question #79497 – Math – Differential Equations

QUESTION

Solve the partial differential equation

$$(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$$

SOLUTION

Let us use some well-known facts.

Let the given differential equation be

$$F(D, D') = f(x, y).$$

Factorize $F(D, D')$ into linear factors. Then use the following results.

Rule I. Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of C.F. is taken as

$$e^{\left(\frac{cx}{b}\right)}\varphi(by + ax), \ \ if\ b \neq 0$$

We now have three particular cases of Rule I:

Rule IA. Take $c = 0$ in Rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is

$$\varphi(by + ax), \ \ if\ b \neq 0.$$

Rule IB. Take $a = 0$ in Rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is

$$e^{\left(\frac{cx}{b}\right)}\varphi(by), \ \ if\ b \neq 0.$$

Rule IC. Take $a = c = 0$ and $b = 1$ in Rule I. Hence corresponding to each linear factor $(1 \cdot D)$, the part of C.F. is

$$\varphi(y).$$

In our case,

$$(D^2+2DD'+D'^2-2D-2D')z=\sin(x+2y)\rightarrow z(x,y)=C.F.+P.I.$$

0 STEP: We factor the expression

$$D^2+2DD'+D'^2-2D-2D'=(D^2+2DD'+D'^2)+(-2D-2D')=$$

$$=(D+D')^2-2\cdot(D+D')=(D+D')(D+D'-2)$$

Conclusion,

$$\boxed{(D^2+2DD'+D'^2-2D-2D')z=(D+D')(D+D'-2)z}$$

1 STEP: Let find C.F.

$$C.F.=(C.F.)_1+(C.F.)_2$$

$$\begin{cases}(D+D')z\\(bD-aD'-c)z\end{cases}\rightarrow\begin{cases}b=1\\a=-1\\c=0\end{cases}\rightarrow(C.F.)_1=e^{\left(\frac{0\cdot x}{1}\right)}\cdot\varphi_1(1\cdot y+(-1)\cdot x)\rightarrow$$

$$\boxed{(C.F.)_1=\varphi_1(y-x), where\ \varphi_1\ is\ an\ arbitrary\ function}$$

$$\begin{cases}(D+D'-2)z\\(bD-aD'-c)z\end{cases}\rightarrow\begin{cases}b=1\\a=-1\\c=2\end{cases}\rightarrow(C.F.)_2=e^{\left(\frac{2\cdot x}{1}\right)}\cdot\varphi_2(1\cdot y+(-1)\cdot x)\rightarrow$$

$$\boxed{(C.F.)_2=e^{2x}\varphi_2(y-x), where\ \varphi_2\ is\ an\ arbitrary\ function}$$

Then,

$$C.F.=(C.F.)_1+(C.F.)_2\rightarrow\boxed{C.F.=\varphi_1(y-x)+e^{2x}\varphi_2(y-x)}$$

2 STEP: Let find P.I.

$$P.I. = \frac{1}{D^2+2DD'+D'^2-2D-2D'}\sin(x+2y) = \frac{1}{D^2+2DD'+D'^2-2D-2D'}\sin(1\cdot x+2\cdot y) =$$

$$= \frac{1}{(1\cdot i)^2+2\cdot(1\cdot i)\cdot(2\cdot i)+(2\cdot i)^2-2D-2D'}\sin(1\cdot x+2\cdot y) =$$

$$= \frac{1}{i^2+2\cdot 2i^2+4i^2-2D-2D'}\sin(x+2y) = \frac{1}{9i^2-2D-2D'}\sin(x+2y) =$$

$$= \frac{1}{-9-2D-2D'}\sin(x+2y) = \frac{1}{-1\cdot(2D+2D'+9)}\sin(x+2y) = \frac{-1}{(2D+2D')+9}\sin(x+2y) =$$

$$= \frac{-1\cdot\big((2D+2D')-9\big)}{\big((2D+2D')+9\big)\cdot\big((2D+2D')-9\big)}\sin(x+2y) = \frac{9-(2D+2D')}{(2D+2D')^2-81}\sin(x+2y) =$$

$$= \frac{9-(2D+2D')}{4D^2+8DD'+4D'^2-81}\sin(x+2y) =$$

$$= \frac{9-(2D+2D')}{4\cdot(1\cdot i)^2+8\cdot(1\cdot i)\cdot(2\cdot i)+4\cdot(2\cdot i)^2-81}\sin(1\cdot x+2\cdot y) =$$

$$= \frac{9-(2D+2D')}{4i^2+16i^2+16i^2-81}\sin(x+2y) = \frac{9-(2D+2D')}{36i^2-81}\sin(x+2y) = \frac{9-(2D+2D')}{-36-81}\sin(x+2y) =$$

$$= \frac{9-(2D+2D')}{-117}\sin(x+2y) = \frac{1}{117}\cdot\big((2D+2D')-9\big)\sin(x+2y) =$$

$$= \frac{1}{117}\left(2\cdot\frac{\partial}{\partial x}+2\cdot\frac{\partial}{\partial y}-9\right)\sin(x+2y) =$$

$$= \frac{1}{117}\left(2\cdot\frac{\partial}{\partial x}(\sin(x+2y))+2\cdot\frac{\partial}{\partial y}(\sin(x+2y))-9\cdot\sin(x+2y)\right) =$$

$$= \frac{1}{117}(2\cdot\cos(x+2y)+2\cdot 2\cdot\cos(x+2y)-9\cdot\sin(x+2y)) = \frac{6\cos(x+2y)-9\sin(x+2y)}{117} =$$

$$= \frac{3\cdot(2\cos(x+2y)-3\sin(x+2y))}{3\cdot 39} = \frac{2\cos(x+2y)-3\sin(x+2y)}{39} \equiv$$

$$\equiv \frac{2}{39}\cdot\cos(x+2y)-\frac{3}{39}\cdot\sin(x+2y)$$

Conclusion,

$$\boxed{P.I. = \frac{2}{39}\cdot\cos(x+2y)-\frac{3}{39}\cdot\sin(x+2y)}$$

General Conclusion,

$$z(x,y) = C.F. + P.I. = \varphi_1(y-x) + e^{2x}\varphi_2(y-x) + \frac{2}{39}\cdot\cos(x+2y) - \frac{3}{39}\cdot\sin(x+2y) \to$$

$$\boxed{z(x,y) = \varphi_1(y-x) + e^{2x}\varphi_2(y-x) + \frac{2}{39}\cdot\cos(x+2y) - \frac{3}{39}\cdot\sin(x+2y)}$$

ANSWER:

$$\begin{cases} z(x,y) = \varphi_1(y-x) + e^{2x}\varphi_2(y-x) + \frac{2}{39}\cdot\cos(x+2y) - \frac{3}{39}\cdot\sin(x+2y) \\ \textit{where } \varphi_1 \textit{ and } \varphi_2 \textit{ are arbitrary functions} \end{cases}$$