Question

Transform the parabolic equations
4Uxx + 12Uxy + 9Uyy − 2Ux + U = 0

A. uξξ −1/3uη +1/9u = −
B. uξξ −1/3uη +1/9u = 0
C. uξξ − uηu=0
D. uξξ−1/5uηξ +1/2u=0

Solution

$$4U_{xx} + 12U_{xy} + 9U_{yy} - 2U_x + U = 0$$

$$a_{11} = 4, a_{12} = 6, a_{22} = 9$$

$$\Delta = a_{12}^2 - a_{11}a_{22} = 36 - 36 = 0$$

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{\Delta}}{a_{11}} = \frac{6}{4} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x + C \Rightarrow C = y - \frac{3}{2}x$$

Let

$$\xi = y - \frac{3}{2}x$$

$$\eta = y + \frac{3}{2}x$$

$$U(x, y) = u(\xi, \eta)$$

Then

$$U_x = u_\xi \xi_x + u_\eta \eta_x = -\frac{3}{2}u_\xi + \frac{3}{2}u_\eta$$

$$U_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta$$

$$U_{xx} = -\frac{3}{2}\left(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x\right) + \frac{3}{2}\left(u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x\right) = \frac{9}{4}u_{\xi\xi} - \frac{9}{2}u_{\eta\xi} + \frac{9}{4}u_{\eta\eta}$$

$$U_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$U_{yx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x = -\frac{3}{2}u_{\xi\xi} + \frac{3}{2}u_{\eta\eta}$$

Then

$$4U_{xx} + 12U_{xy} + 9U_{yy} - 2U_x + U = 0 \Rightarrow 4\left(\frac{9}{4}u_{\xi\xi} - \frac{9}{2}u_{\eta\xi} + \frac{9}{4}u_{\eta\eta}\right) + 12\left(-\frac{3}{2}u_{\xi\xi} + \frac{3}{2}u_{\eta\eta}\right) +$$

$$+9\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) - 2\left(-\frac{3}{2}u_\xi + \frac{3}{2}u_\eta\right) + u = 0 \Rightarrow$$

$$\Rightarrow u_{\xi\xi}(9 - 18 + 9) + u_{\eta\eta}(9 + 18 + 9) + u_{\eta\xi}(-18 + 18) + 3u_\xi - 3u_\eta + u = 0 \Rightarrow$$

$$\Rightarrow 36u_{\eta\eta} = 3u_\eta - 3u_\xi - u \Rightarrow u_{\eta\eta} = \frac{3u_\eta - 3u_\xi - u}{36}$$