ANSWER on Question #79371 – Math – Differential Equations

QUESTION

Solve the differential equation

$$\frac{dy}{dx} = 2y^2 + 3xy/x^2$$

Possible answers:

a)

$$y = \frac{cx^3}{1 - cx^2}$$

b)

$$y = cx31 + cx2$$

c)

$$y = -cx31 - cx2$$

d)

$$y = -cx31 + cx2$$

SOLUTION

Hint: Since only the version of the answer (a) is written with the help of Latex functions, only its formula editor understands uniquely, only I will consider it correct from the whole question.

Hint: Equation Editor predetermined equation may be understood in two ways

1)

$$\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2}$$

2)

$$\frac{dy}{dx} = 2y^2 + \frac{3xy}{x^2}$$

We will solve each of these equations separately.

Hint: Most likely, in the variants of the answer some function signs are omitted. Probably, the answer options should have looked like this

a)

$$y = \frac{cx^3}{1 - cx^2}$$

b)

$$y = \frac{cx^3}{1 + cx^2}$$

c)

$$y = -\frac{cx^3}{1 - cx^2}$$

d)

$$y = -\frac{cx^3}{1 + cx^2}$$

Now we come to the solution of this equation.

1 CASE:

$$\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2} \rightarrow \frac{dy}{dx} = \frac{2y^2}{x^2} + \frac{3xy}{x^2} \rightarrow \boxed{\frac{dy}{dx} = 2 \cdot \left(\frac{y}{x}\right)^2 + 3 \cdot \frac{y}{x}}$$

We introduce the substitution

$$u = \frac{y}{x} \rightarrow y = ux \rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u \cdot 1 \rightarrow \boxed{\frac{dy}{dx} = \frac{du}{dx} \cdot x + u}$$

Then

$$\begin{cases} \dfrac{dy}{dx} = 2 \cdot \left(\dfrac{y}{x}\right)^2 + 3 \cdot \dfrac{y}{x} \\ u = \dfrac{y}{x} \\ \dfrac{dy}{dx} = \dfrac{du}{dx} \cdot x + u \end{cases} \rightarrow \frac{du}{dx} \cdot x + u = 2u^2 + 3u \rightarrow \frac{du}{dx} \cdot x = 2u^2 + 3u - u \rightarrow \frac{du}{dx} \cdot x$$

$$= 2u^2 + 2u \rightarrow$$

$$\frac{du}{dx} \cdot x = 2u^2 + 2u \Big| \times \left(\frac{2 \cdot dx}{x \cdot (2u^2 + 2u)}\right) \rightarrow \frac{2 \cdot du}{2u^2 + 2u} = \frac{2 \cdot dx}{x} \rightarrow \frac{2 \cdot \big((u+1) - u\big)du}{2u(u+1)}$$

$$= \frac{2 \cdot dx}{x} \rightarrow$$

$$\left(\frac{1}{u} - \frac{1}{u+1}\right) du = \left(\frac{2}{x}\right) dx \rightarrow \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = \int \left(\frac{2}{x}\right) dx \rightarrow \ln|u| - \ln|u+1|$$

$$= 2 \cdot \ln|x| + \ln|c| \rightarrow$$

$$\ln\left|\frac{u}{u+1}\right| = \ln|cx^2| \rightarrow \frac{u}{u+1} = cx^2 \Big| \times (u+1) \rightarrow u = cx^2 \cdot (u+1) \rightarrow u = cx^2 \cdot u + cx^2 \rightarrow$$

$$u - cx^2 \cdot u = cx^2 \rightarrow u(1 - cx^2) = cx^2 \rightarrow \boxed{u = \frac{cx^2}{1 - cx^2}}$$

We recall that we introduced a substitution

$$u = \frac{y}{x}$$

Then,

$$\begin{cases} u = \dfrac{cx^2}{1 - cx^2} \\ u = \dfrac{y}{x} \end{cases} \rightarrow \frac{y}{x} = \frac{cx^2}{1 - cx^2} \Big| \times (x) \rightarrow y = \frac{cx^3}{1 - cx^2}$$

Conclusion,

$$\boxed{\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2} \rightarrow y = \frac{cx^3}{1 - cx^2}}$$

2 CASE:

$$\frac{dy}{dx} = 2y^2 + \frac{3xy}{x^2} \to \frac{dy}{dx} = 2y^2 + 3 \cdot \frac{y}{x}$$

As we can see, this is the Bernoulli equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

(More information: https://en.wikipedia.org/wiki/Bernoulli_differential_equation)

In our case,

$$\frac{dy}{dx} = 2y^2 + 3 \cdot \frac{y}{x}\Big| \div (y^2) \to \frac{1}{y^2} \cdot \frac{dy}{dx} = 2 + \frac{3}{x} \cdot \frac{1}{y}$$

We introduce the substitution

$$u = \frac{1}{y} \to \frac{du}{dx} = -\frac{1}{y^2} \cdot \frac{dy}{dx} \to \boxed{\begin{cases} y = \frac{1}{u} \\ \frac{1}{y^2} \cdot \frac{dy}{dx} = -\frac{du}{dx} \end{cases}}$$

Then,

$$\begin{cases} \frac{1}{y^2} \cdot \frac{dy}{dx} = 2 + \frac{3}{x} \cdot \frac{1}{y} \\ y = \frac{1}{u} \\ \frac{1}{y^2} \cdot \frac{dy}{dx} = -\frac{du}{dx} \end{cases} \to \boxed{-\frac{du}{dx} = 2 + \frac{3}{x} \cdot u}$$

We have obtained a nonhomogeneous equation of the first order. We solve it by the method of variation of the constant.

1 STEP: Solve a homogeneous equation.

$$-\frac{du}{dx} = \frac{3}{x} \cdot u\Big| \times \left(\frac{-dx}{u}\right) \to \frac{du}{u} = \frac{-3 \cdot dx}{x} \to \int \left(\frac{1}{u}\right) du = \int \left(-\frac{3}{x}\right) dx \to \ln|u|$$
$$= -3 \cdot \ln|x| + \ln|c| \to$$

$$\ln|u| = \ln\left|\frac{c}{x^3}\right| \to \boxed{u = \frac{c}{x^3}}$$

2 STEP: Variations of the constant.

We will assume that $c = c(x)$. Then,

$$u = \frac{c(x)}{x^3} \to \boxed{\frac{du}{dx} = \frac{1}{x^3} \cdot \frac{dc}{dx} - \frac{3c(x)}{x^4}}$$

We substitute the obtained formula for the derivative in the initial equation

$$\begin{cases} -\dfrac{du}{dx} = 2 + \dfrac{3}{x} \cdot u \\ u = \dfrac{c(x)}{x^3} \\ \dfrac{du}{dx} = \dfrac{1}{x^3} \cdot \dfrac{dc}{dx} - \dfrac{3c}{x^4} \end{cases} \to -\left(\frac{1}{x^3} \cdot \frac{dc}{dx} - \frac{3c(x)}{x^4}\right) = 2 + \frac{3}{x} \cdot \frac{c(x)}{x^3} \to$$

$$-\frac{1}{x^3} \cdot \frac{dc}{dx} + \frac{3c(x)}{x^4} = 2 + \frac{3c(x)}{x^4} \to -\frac{1}{x^3} \cdot \frac{dc}{dx} = 2 \Big| \times (-x^3 dx) \to dc = (-2x^3)dx \to$$

$$\int dc = \int (-2x^3)dx \to c(x) = -2 \cdot \frac{x^{3+1}}{3+1} + C \to c(x) = C - \frac{x^4}{2} = \frac{2C - x^4}{2} \equiv \frac{A - x^4}{2}$$

Then,

$$\begin{cases} u = \dfrac{c(x)}{x^3} \\ c(x) = \dfrac{A - x^4}{2} \end{cases} \to \boxed{u = \frac{A - x^4}{2x^3}}$$

We recall that we introduced a substitution

$$u = \frac{1}{y}$$

Then

$$\begin{cases} u = \frac{A - x^4}{2x^3} \\ u = \frac{1}{y} \end{cases} \rightarrow \frac{1}{y} = \frac{A - x^4}{2x^3} \rightarrow y = \frac{2x^3}{A - x^4}$$

Conclusion,

$$\boxed{\frac{dy}{dx} = 2y^2 + \frac{3xy}{x^2} \rightarrow y = \frac{2x^3}{A - x^4}}$$

ANSWER:

If the initial equation was

$$\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2},$$

then the solution is

$$a)\quad y = \frac{cx^3}{1 - cx^2}$$

If the initial equation was

$$\frac{dy}{dx} = 2y^2 + \frac{3xy}{x^2}$$

then

$$\textit{In the provided options there is no correct answer}$$

$$\frac{dy}{dx} = 2y^2 + \frac{3xy}{x^2} \rightarrow y = \frac{2x^3}{A - x^4} \textit{ will be the correct answer.}$$