Answer on question #79319

Show that $1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}} \geq \sqrt{2(n-1)}$ for $n \in \mathbb{N}, n>1$

Solve using laws and theorem of inequalities

Solution.

Write the inequality for the harmonic and quadratic mean of numbers $1, \sqrt{2}, ..., \sqrt{n}$:

$$\frac{n}{1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}} \leq \sqrt{\frac{1+2+...+n}{n}}$$

Using the formula for the sum of the first n numbers:

$$\frac{n}{1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}} \leq \sqrt{\frac{n(n+1)}{2n}}$$

or

$$\frac{n}{1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}} \leq \sqrt{\frac{n+1}{2}}$$

As $1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}$ and $\sqrt{\frac{n+1}{2}}$ are both greater than 0 we can write:

$$\frac{n}{\sqrt{\frac{n+1}{2}}} \leq 1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}$$

or

$$1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}} \geq n\sqrt{\frac{2}{n+1}} = \sqrt{\frac{2n^2}{n+1}} \geq \sqrt{\frac{2(n^2-1)}{n+1}} = \sqrt{2(n-1)}$$

Therefore

$$1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}} \geq \sqrt{2(n-1)}$$