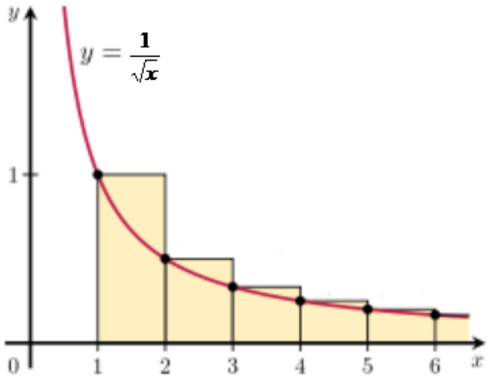
Answer on Question #79315 – Math – Algebra

Question

Show that

$$1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}\geq\sqrt{2(n-1)} \text{ for } n\in N,\ n>1.$$

Solution



The function $y=\frac{1}{\sqrt{x}}$ decreases on $[1,n+1]$. Then Left Riemann sum of $y=\frac{1}{\sqrt{x}}$ on $[1,n+1]$ is greater than Riemann integral of $y=\frac{1}{\sqrt{x}}$ on $[1,n+1]$.

Since Left Riemann sum of $y=\frac{1}{\sqrt{x}}$ on $[1,n+1]$ is equal

$$1\cdot1+\frac{1}{\sqrt{2}}\cdot1+...+\frac{1}{\sqrt{n}}\cdot1=1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}},$$

for $n>1$ we get

$$1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}\geq\int_1^{n+1}\frac{1}{\sqrt{x}}dx=2\sqrt{x}\Big|_1^{n+1}=2\sqrt{n+1}-2\sqrt{1}=2\sqrt{n+1}-2.$$

Then we need to verify the following inequality

$$2\sqrt{n+1}-2\geq\sqrt{2(n-1)},$$

$$4(n+1)+4-8\sqrt{n+1}\geq2(n-1),$$

$$4n+8-8\sqrt{n+1}\geq2n-2,$$

$$2n+10 \ge 8\sqrt{n+1},$$

$$4n^2+40n+100 \ge 64(n+1),$$

$$4n^2-24n+36 \ge 0,$$

$$n^2-6n+9 \ge 0,$$

$$(n-3)^2 \ge 0.$$

Therefore, $2\sqrt{n+1}-2 \ge \sqrt{2(n-1)}$ is true for all $n \ge 1$. Since $1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}} \ge 2\sqrt{n+1}-2$ for $n>1$ and $2\sqrt{n+1}-2 \ge \sqrt{2(n-1)}$ for all $n \ge 1$,

$$1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}} \ge \sqrt{2(n-1)} \text{ for all } n > 1.$$