ANSWER on Question #79307 – Math – Differential Equations

QUESTION

Exact equations

$$\left(e^{x^2}y\right)(1+2(x^2)y)dx+(x^3)\left(e^{x^2}y\right)dy=0$$

Integrating factor:

1. Differential equation

$$(2xy)\ dx+(2(x^2)+3)dy=0$$

Having an integration factor is just a function of y.

2. Differential equation

$$((x^2)+3x+2)dx+((x^2)+x+1)dy=0$$

Having an integration factor is just a function of $(x+y)$.

SOLUTION

The first part: solve the above differential equation.

1 STEP: We transform this differential equation

$$\left(e^{x^2}y\right)(1+2(x^2)y)dx+(x^3)\left(e^{x^2}y\right)dy=0\to$$

$$\left(e^{x^2}y\right)(1+2(x^2)y)dx=-(x^3)\left(e^{x^2}y\right)dy\Big|\div\left(\frac{-1}{dx\cdot(x^3)\left(e^{x^2}y\right)}\right)\to$$

$$-\frac{\left(e^{x^2}y\right)(1+2(x^2)y)dx}{dx\cdot(x^3)\left(e^{x^2}y\right)}=\frac{(x^3)\left(e^{x^2}y\right)dy}{dx\cdot(x^3)\left(e^{x^2}y\right)}\to-\frac{(1+2(x^2)y)}{x^3}=\frac{dy}{dx}\to$$

$$\frac{dy}{dx}=-\frac{1}{x^3}-\frac{2yx^2}{x^3}\to\frac{dy}{dx}=-\frac{1}{x^3}-\frac{2y}{x}\to\boxed{\frac{dy}{dx}+\frac{2y}{x}=-\frac{1}{x^3}}$$

Conclusion,

$$\left(e^{x^2}y\right)(1+2(x^2)y)dx+(x^3)\left(e^{x^2}y\right)dy=0\to\frac{dy}{dx}+\frac{2y}{x}=-\frac{1}{x^3}$$

Nonhomogeneous differential equation of the first order.

2 STEP: Let us solve the transformed equation.

Since,

$$\frac{dy}{dx}+\frac{2y}{x}=-\frac{1}{x^3}$$

Nonhomogeneous differential equation of the first order, then the solution consists of two parts:

$$y(x)=y_1(x)+y_2(x),\ \ where\ \begin{cases} y_1(x)-solution\ of\ the\ homogeneous\ equation \\ y_2(x)-a\ particular\ solution\ of\ the\ nonhomogeneous\ equation\end{cases}$$

2A STEP: We solve the homogeneous equation.

For the solution, we use the method of separation of variables.

(More information: https://en.wikipedia.org/wiki/Separation_of_variables)

$$\frac{dy}{dx}+\frac{2y}{x}=0\rightarrow\frac{dy}{dx}=-\frac{2y}{x}\rightarrow\frac{dy}{y}=-2\frac{dx}{x}\rightarrow\int\frac{dy}{y}=\int\left(-2\frac{dx}{x}\right)\rightarrow\ln|y|=-2\cdot\ln|x|+\ln|C|\rightarrow$$

$$\ln|y|=\ln|x^{-2}|+\ln|C|\rightarrow\ln|y|=\ln|C\cdot x^{-2}|\rightarrow y_1(x)=\frac{C}{x^2}$$

Conclusion,

$$\boxed{\frac{dy}{dx}+\frac{2y}{x}=0\rightarrow y_1(x)=\frac{C}{x^2}}$$

2B STEP: We solve the nonhomogeneous equation.

For the solution, we use the method of variation of the parameter.

(More information: https://en.wikipedia.org/wiki/Variation_of_parameters)

$$\begin{cases}\frac{dy}{dx}+\frac{2y}{x}=-\frac{1}{x^3} \\ y(x)=\frac{C(x)}{x^2}\rightarrow\frac{dy}{dx}=\frac{1}{x^2}\cdot\frac{dC}{dx}-\frac{2C(x)}{x^2}\end{cases}\rightarrow\frac{1}{x^2}\cdot\frac{dC}{dx}-\frac{2C(x)}{x^2}+\frac{2}{x}\cdot\frac{C(x)}{x^2}=-\frac{1}{x^3}\rightarrow$$

$$\frac{1}{x^2}\cdot\frac{dC}{dx}=-\frac{1}{x^3}\bigg|\times(dx\cdot x^2)\rightarrow dC=-\frac{dx}{x}\rightarrow\int dC=\int\left(-\frac{dx}{x}\right)\rightarrow\boxed{C(x)=-\ln|x|+C_1}$$

Then,

$$\begin{cases} y(x) = \dfrac{C(x)}{x^2} \\ C(x) = C_1 - \ln|x| \end{cases} \rightarrow y(x) = \frac{C_1 - \ln|x|}{x^2} = \frac{C_1}{x^2} - \frac{\ln|x|}{x}, \ \ where \ \begin{cases} y_1(x) = \dfrac{C_1}{x^2} \\ y_2(x) = -\dfrac{\ln|x|}{x} \end{cases}$$

Conclusion,

$$\boxed{\left(e^{x^2}y\right)(1 + 2(x^2)y)dx + (x^3)\left(e^{x^2}y\right)dy = 0 \rightarrow y(x) = \frac{C_1}{x^2} - \frac{\ln|x|}{x}}$$

The second part: we solve the problems associated with the integral factor.

We recall the definition of an equation in exact differentials:

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Recall another definition:

If the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is not exact, but the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact, then $\mu(x,y)$ is an integrating factor of equation.

We recall one theorem connected with the integral factor.

Theorem:

If

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} \text{ is continuous and depends only on } x, \text{then}$$

$$\mu(x) = \exp\left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) dx\right] \text{ is an integrating factor for the DE.}$$

If

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} \text{ is continuous and depends only on } y, \text{then}$$

$$\mu(y) = \exp\left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right) dy\right] \text{ is an integrating factor for the DE.}$$

Now we can start solving the problem.

1. Differential equation

$$(2xy)\,dx + (2(x^2) + 3)dy = 0$$

Having an integration factor is just a function of y.

In our case,

$$\begin{cases} M(x,y) = 2xy \\ N(x,y) = 2x^2 + 3 \end{cases} \rightarrow \begin{cases} \dfrac{\partial M}{\partial y} = 2x \\ \dfrac{\partial N}{\partial x} = 2 \cdot 2x = 4x \end{cases} \rightarrow \frac{\partial M}{\partial y} = 2x \neq 4x = \frac{\partial N}{\partial x}$$

Conclusion,

$$\boxed{(2xy)\,dx + (2(x^2) + 3)dy = 0 \; is\ not\ exact}$$

We now verify the fulfillment of the conditions of the theorem:

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{4x - 2x}{2xy} = \frac{2x}{2xy} = \frac{1}{y} \; is\ continuous\ and\ depends\ only\ on\ y$$

Then,

$$\exists \mu(y)\ is\ an\ integrating\ factor\ for\ the\ DE.$$

We can easily find this integral factor, using the formula from the theorem

$$\mu(y) = \exp\left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right) dy\right] = \exp\left[\int \left(\frac{1}{y}\right) dy\right] = \exp(\ln y) = y \rightarrow \boxed{\mu(y) = y}$$

Checking:

$$(2xy)\,dx + (2(x^2) + 3)dy = 0| \times (y) \rightarrow (2xy^2)\,dx + (2yx^2 + 3y)dy = 0 \rightarrow$$

$$\begin{cases} M(x,y) = 2xy^2 \\ N(x,y) = (2x^2y + 3y) \end{cases} \rightarrow \begin{cases} \dfrac{\partial M}{\partial y} = 4xy \\ \dfrac{\partial N}{\partial x} = 2 \cdot 2xy = 4xy \end{cases} \rightarrow \frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}$$

Conclusion,

$$\boxed{(2xy^2)\,dx + (2yx^2 + 3y)dy = 0 \; is\ exact}$$

2. Differential equation

$$((x^2)+3x+2)dx+((x^2)+x+1)dy=0$$

Having an integration factor is just a function of $(x+y)$.

In our case,

$$\begin{cases} M(x,y)=x^2+3x+2 \\ N(x,y)=x^2+x+1 \end{cases} \rightarrow \begin{cases} \dfrac{\partial M}{\partial y}=0 \\ \dfrac{\partial N}{\partial x}=2x+1 \end{cases} \rightarrow \frac{\partial M}{\partial y}=0 \neq 2x+1=\frac{\partial N}{\partial x}$$

Conclusion,

$$\boxed{\big((x^2)+3x+2\big)dx+\big((x^2)+x+1\big)dy=0 \ is\ not\ exact}$$

We now verify the fulfillment of the conditions of the theorem:

$$\frac{\partial M/\partial y-\partial N/\partial x}{N}=\frac{0-(2x+1)}{x^2+x+1}=-\frac{2x+1}{x^2+x+1}\ is\ continuous\ and\ depends\ only\ on\ x$$

Then,

$$\exists\mu(x)\ is\ an\ integrating\ factor\ for\ the\ DE.$$

We can easily find this integral factor, using the formula from the theorem

$$\mu(x)=\exp\left[\int\left(\frac{\partial M/\partial y-\partial N/\partial x}{N}\right)dx\right]=\exp\left[\int\left(-\frac{2x+1}{x^2+x+1}\right)dx\right]=\begin{bmatrix} x^2+x+1=t \\ (2x+1)dx=dt \end{bmatrix}=$$

$$=\exp\left[\int\left(-\frac{1}{t}\right)dt\right]=\exp[-\ln|t|]=\exp\left[\ln\left|\frac{1}{t}\right|\right]=\frac{1}{t}=\frac{1}{x^2+x+1}\rightarrow\boxed{\mu(x)=\frac{1}{x^2+x+1}}$$

Checking:

$$\left((x^2)+3x+2\right)dx+\left((x^2)+x+1\right)dy=0\Big|\times\left(\frac{1}{x^2+x+1}\right)\rightarrow$$

$$\frac{(x^2+3x+2)}{x^2+x+1}dx+\frac{(x^2+x+1)}{x^2+x+1}dy=0\rightarrow\left(\frac{x^2+3x+2}{x^2+x+1}\right)\cdot dx+1\cdot dy=0$$

$$\begin{cases}M(x,y)=\dfrac{x^2+3x+2}{x^2+x+1}\\ N(x,y)=1\end{cases}\rightarrow\begin{cases}\dfrac{\partial M}{\partial y}=0\\ \dfrac{\partial N}{\partial x}=0\end{cases}\rightarrow\frac{\partial M}{\partial y}=0=\frac{\partial N}{\partial x}$$

Conclusion,

$$\boxed{\left(\frac{x^2+3x+2}{x^2+x+1}\right)\cdot dx+1\cdot dy=0\ is\ exact}$$

ANSWER:

$$\left(e^{x^2}y\right)(1+2(x^2)y)dx+(x^3)\left(e^{x^2}y\right)dy=0\rightarrow y(x)=\frac{C_1}{x^2}-\frac{\ln|x|}{x}$$

1.

$$\mu(y)=y\ is\ an\ integrating\ factor\ for\ the\ DE$$

$$\mu(y)=y\ is\ a\ function\ only\ of\ the\ variable\ y$$

2.

$$\mu(x)=x^2+x+1\ is\ an\ integrating\ factor\ for\ the\ DE$$

$$\mu(x)=x^2+x+1\ is\ not\ a\ function\ of\ a\ variable\ (x+y)$$