Answer on Question #79242 - Math - Calculus

Show that $1+1/\sqrt{2}+...+1/\sqrt{n} \geq \sqrt{2(n-1)}$ for $n \in N$, $n > 1$

The answer:

Let us cosider the equality:

$$\sqrt{k+1}-\sqrt{k} = (\sqrt{k+1}-\sqrt{k})\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}} = \frac{1}{\sqrt{k+1}+\sqrt{k}} < \frac{1}{2\sqrt{k}}$$

for any $k > 0$.

So for

$$\begin{aligned}
k &= 1 \quad \Rightarrow \quad \frac{1}{2} > \sqrt{2}-\sqrt{1} \\
k &= 2 \quad \Rightarrow \quad \frac{1}{2\sqrt{2}} > \sqrt{3}-\sqrt{2} \\
k &= 3 \quad \Rightarrow \quad \frac{1}{2\sqrt{3}} > \sqrt{4}-\sqrt{3} \\
& \quad ... \\
k &= n-1 \quad \Rightarrow \quad \frac{1}{2\sqrt{n-1}} > \sqrt{n}-\sqrt{n-1} \\
k &= n \quad \Rightarrow \quad \frac{1}{2\sqrt{n}} > \sqrt{n+1}-\sqrt{n}
\end{aligned}$$

.

Summarizing the last equation set one gets

$$\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}}\right) > \sqrt{n+1}-1$$

so

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1).$$

Let us demonstrate $2(\sqrt{n+1}-1) \geq \sqrt{2(n-1)}$.

For $n > 1$, i.e. both LHS and RHS are positive, we can squared

$$\begin{aligned}
2(\sqrt{n+1}-1) &\geq \sqrt{2(n-1)} \\
2(n+1+1-2\sqrt{n+1}) &\geq (n-1) \\
n+1-4\sqrt{n+1}+4 &= (\sqrt{n+1}-2)^2 > 0
\end{aligned}$$

So

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1) \geq \sqrt{2(n-1)}$$