ANSWER on Question #79218 – Math – Linear Algebra

QUESTION

We know that the set $F(\mathbb{R})$ of functions $f : \mathbb{R} \to \mathbb{R}$, together with pointwise addition and scalar multiplication

$$(f+g)(x) = f(x) + g(x)\ for\ all\ f, g \in F(\mathbb{R})\ and\ x\ \in \mathbb{R}$$

$$(\lambda \cdot f)(x) = \lambda f(x)\ for\ all\ f\ \in F(\mathbb{R}), \lambda \in \mathbb{R}\ and\ x \in \mathbb{R}.$$

In this problem, you are asked to discuss whether $F(\mathbb{R})$ continues to be a vector space when the operations $(+,\cdot)$ are replaced by other addition/scalar multiplication operations $(+', \cdot')$.

a) Suppose we define $+'$ and $\cdot'$ on $F(\mathbb{R})$ by

$$(f+'g)(x) = f(x) - g(x)\ for\ all\ f, g \in F(\mathbb{R})\ and\ x \in \mathbb{R}$$

$$(\lambda \cdot' f)(x) = f(\lambda x)\ for\ all\ f\ \in F(\mathbb{R}), \lambda \in \mathbb{R}\ and\ x \in \mathbb{R}.$$

With these operations, is the $(F(\mathbb{R}), +', \cdot')$ a vector space? Why or why not?

b) Suppose that we now define $+'$ and $\cdot'$ on $F(\mathbb{R})$ by

$$(f+'g)(x) = f\big(g(x)\big)\ for\ all\ f, g \in F(\mathbb{R})\ and\ x \in \mathbb{R}$$

$$(\lambda \cdot' f)(x) = \lambda f(x)\ for\ all\ f\ \in F(\mathbb{R}), \lambda \in \mathbb{R}\ and\ x \in \mathbb{R}.$$

(i.e., $\cdot'$ is the usual scalar multiplication). With these operations, is the $(F(\mathbb{R}), +', \cdot')$ a vector space? Why or why not?

SOLUTION

A vector space over a field F is a set V together with two operations that satisfy the eight axioms listed below.

1) The first operation, called **vector addition** or simply **addition** $+: V \times V \rightarrow V$, takes any two vectors v and w and assigns to them a third vector which is commonly written as $v + w$, and called the sum of these two vectors. (Note that the resultant vector is also an element of the set V).

2) The second operation, called scalar multiplication $\cdot: F \times V \rightarrow V$, takes any scalar α and any vector v and gives another vector αv. (Similarly, the vector αv is an element of the set V).

Elements of V are commonly called *vectors*. Elements of F are commonly called *scalars*.

In the two examples above, the field is the field of the real numbers and the set of the vectors consists of the planar arrows with fixed starting point and of pairs of real numbers, respectively.

To qualify as a vector space, the set V and the operations of addition and multiplication must adhere to a number of requirements called axioms. In the list below, let u, v and w be arbitrary vectors in V, and α and β scalars in F.

We recall the axioms of a vector space:

1) Associativity of addition

$$u + (v + w) = (u + v) + w$$

2) Commutativity of addition

$$u + v = v + u$$

3) Identity element of addition

$$There\ exists\ an\ element\ \mathbf{0} \in V, called\ the\ zero\ vector, such\ that\ v + \mathbf{0} = v\ for\ all\ v \in V.$$

4) Inverse element of addition

$$For\ every\ v \in V, there\ exists\ an\ elemet\ (-v) \in V, called\ the\ additive\ inverse\ of\ v, such\ that$$

$$v + (-v) = \mathbf{0}$$

5) Compatibility of scalar multiplication with field multiplication

$$\alpha(\beta v) = (\alpha\beta)v$$

6) Identity element of scalar multiplication

$$1v,\quad where\ 1\ denotes\ the\ multiplicative\ identity\ in\ F$$

7) Distributivity of scalar multiplication with respect to vector addition

$$\alpha(u+v)=\alpha u+\alpha v$$

8) Distributivity of scalar multiplication with respect to field addition

$$(\alpha+\beta)v=\alpha u+\beta u$$

(More information: https://en.wikipedia.org/wiki/Vector_space)

If space $(F(\mathbb{R}),+',\cdot')$ does not satisfy at least one of the axioms, then it is not a vector space.

Our solution is as follows: we will check the fulfillment of the axioms from the list. If at least one is not satisfied, then we stop working and conclude that the given space is not a vector space.

a) Suppose we define $+'$ and $\cdot'$ on $F(\mathbb{R})$ by

$$(f+'g)(x)=f(x)-g(x)\ for\ all\ f,g\in F(\mathbb{R})\ and\ x\in\mathbb{R}$$

$$(\lambda\cdot' f)(x)=f(\lambda x)\ for\ all\ f\in F(\mathbb{R}),\lambda\in\mathbb{R}\ and\ x\in\mathbb{R}.$$

With these operations, is the $(F(\mathbb{R}),+',\cdot')$ a vector space? Why of why not?

Axiom 1 (Associativity of addition)

$$\big((f+'g)+'h\big)(x)=(f+'g)(x)-h(x)=f(x)-g(x)-h(x)$$

$$\big(f+'(g+'h)\big)(x)=f(x)-(g+'h)(x)=f(x)-\big(g(x)-h(x)\big)=f(x)-g(x)+h(x)$$

Conclusion,

$$\begin{cases}\big((f+'g)+'h\big)(x)=f(x)-g(x)-h(x)\\ \big(f+'(g+'h)\big)(x)=f(x)-g(x)+h(x)\end{cases}\rightarrow\boxed{\big((f+'g)+'h\big)(x)\neq\big(f+'(g+'h)\big)(x)}$$

Conclusion,

$$\boxed{(F(\mathbb{R}),+',\cdot')\ is\ not\ a\ vector\ space. The\ axiom\ Associativity\ of\ addition\ does\ not\ hold}$$

b) Suppose that we now define $+'$ and $\cdot\,'$ on $F(\mathbb{R})$ by

$$(f+'g)(x) = f\big(g(x)\big)\ for\ all\ f,g \in F(\mathbb{R})\ and\ x \in \mathbb{R}$$

$$(\lambda \cdot' f)(x) = \lambda f(x)\ for\ all\ f \in F(\mathbb{R}), \lambda \in \mathbb{R}\ and\ x \in \mathbb{R}.$$

(i.e., $\cdot\,'$ is the usual scalar multiplication). With these operations, is the $(F(\mathbb{R}), +', \cdot')$ a vector space? Why of why not?

Axiom 1 (Associativity of addition)

$$\big((f+'g)+'h\big)(x) = (f+'g)\big(h(x)\big) = f\Big(g\big(h(x)\big)\Big)$$

$$\big(f+'(g+'h)\big)(x) = f\big((g+'h)(x)\big) = f\Big(g\big(h(x)\big)\Big)$$

Conclusion,

$$\begin{cases} \big((f+'g)+'h\big)(x) = f\Big(g\big(h(x)\big)\Big) \\ \big(f+'(g+'h)\big)(x) = f\Big(g\big(h(x)\big)\Big) \end{cases} \rightarrow \boxed{\big((f+'g)+'h\big)(x) = \big(f+'(g+'h)\big)(x)}$$

Axiom 2 (Commutativity of addition)

$$(f+'g)(x) = f\big(g(x)\big)$$

$$(g+'f)(x) = g\big(f(x)\big)$$

Conclusion,

$$\begin{cases} (f+'g)(x) = f\big(g(x)\big) \\ (g+'f)(x) = g\big(f(x)\big) \end{cases} \rightarrow \boxed{(f+'g)(x) \neq (g+'f)(x)}$$

Conclusion,

$$\boxed{(F(\mathbb{R}), +', \cdot')\ is\ not\ a\ vector\ space. The\ axiom\ Commutativity\ of\ addition\ does\ not\ hold}$$

ANSWER:

a)

$$(F(\mathbb{R}),+',\cdot') \ is\ not\ a\ vector\ space. The\ axiom\ Associativity\ of\ addition\ does\ not\ hold$$

b)

$$(F(\mathbb{R}),+',\cdot') \ is\ not\ a\ vector\ space. The\ axiom\ Commutativity\ of\ addition\ does\ not\ hold$$