Answer on Question #79080 – Math – Calculus

Question

Integrate

$$\int \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}dx$$

Solution

Substitution: $u=x^2 \Rightarrow du=2xdx$

Then

$$\int \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}dx=\frac{1}{2}\int \sqrt{\frac{a^2-u}{a^2+u}}du$$

Substitution:

$$\frac{a^2-u}{a^2+u}=s \Rightarrow ds=\left(-\frac{a^2-u}{\left(a^2+u\right)^2}-\frac{1}{a^2+u}\right)du=\frac{u-a^2-a^2-u}{\left(a^2+u\right)^2}du=\frac{-2a^2}{\left(a^2+u\right)^2}du$$

$$\frac{1}{2}\int \sqrt{\frac{a^2-u}{a^2+u}}du=\frac{1}{2}\int \sqrt{\frac{a^2-u}{a^2+u}}\left(\frac{-2a^2}{\left(a^2+u\right)^2}\right)\cdot\frac{\left(a^2+u\right)^2}{-2a^2}du$$

$$-s-1=\frac{u-a^2}{a^2+u}-1=\frac{u-a^2-a^2-u}{a^2+u}=\frac{-2a^2}{a^2+u}\Rightarrow(-s-1)^2=\frac{\left(-2a^2\right)^2}{\left(a^2+u\right)^2}$$

Then

$$\frac{1}{2}\int \sqrt{\frac{a^2-u}{a^2+u}}\left(\frac{-2a^2}{\left(a^2+u\right)^2}\right)\cdot\frac{\left(a^2+u\right)^2}{-2a^2}du=\frac{1}{2}\int \sqrt{\frac{a^2-u}{a^2+u}}\left(\frac{-2a^2}{\left(a^2+u\right)^2}\right)\cdot\frac{\left(a^2+u\right)^2}{\left(-2a^2\right)^2}(-2a^2)du=$$

$$=-a^2\int \frac{\sqrt{s}}{(-s-1)^2}ds$$

Substitution:

$$p=\sqrt{s}\Rightarrow dp=\frac{1}{2\sqrt{s}}ds$$

Then

$$-a^2\int\frac{\sqrt{s}}{(-s-1)^2}ds=-2a^2\int\frac{s}{2\sqrt{s}(-s-1)^2}ds=-2a^2\int\frac{p^2}{(-p^2-1)^2}dp=-2a^2\int\frac{p^2}{(p^2+1)^2}dp=$$

$$=-2a^2\left(\int\frac{dp}{p^2+1}-\int\frac{dp}{(p^2+1)^2}\right)=-2a^2 arctgp+2a^2\int\frac{dp}{(p^2+1)^2}$$

Consider $\int\frac{dp}{(p^2+1)^2}$

Substitution: $p=\tan w\Rightarrow dp=\frac{1}{\cos^2 w}dw$

Then

$$\int\frac{dp}{(p^2+1)^2}=\int\frac{dw}{\cos^2 w(\tan^2 w+1)^2}=\int\cos^2 w dw=\frac{1}{4}\int(1+\cos 2w)d2w=$$

$$=\frac{w}{2}+\frac{1}{4}\sin 2w+C_1=\frac{\arctan p}{2}+\frac{1}{4}\sin(2\arctan p)+C_1$$

Then

$$-2a^2\arctan p+2a^2\int\frac{dp}{(p^2+1)^2}=-2a^2\arctan p+a^2\arctan p+\frac{a^2}{2}\sin(2\arctan p)+C=$$

$$=-a^2\arctan\sqrt{s}+\frac{a^2}{2}\sin(2\arctan\sqrt{s})+C=$$

$$=-a^2\arctan\sqrt{\frac{a^2-u}{a^2+u}}+\frac{a^2}{2}\sin(2\arctan\sqrt{\frac{a^2-u}{a^2+u}})+C=$$

$$=-a^2\arctan\sqrt{\frac{a^2-x^2}{a^2+x^2}}+\frac{a^2}{2}\sin(2\arctan\sqrt{\frac{a^2-x^2}{a^2+x^2}})+C$$