Answer on Question #79068 – Math – Calculus

Question

If $y = e^{m\tan^{-1}(x)}$, show that $(1+x^2)y_{n+1} + 2(nx-m)y_n + n(n-1)y_{n-1} = 0$

Solution

Leibnitz Theorem

$$\frac{d^n}{dx^n}(uv) = u_n v + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + \cdots + \binom{n}{r}u_{n-r}v_r + \cdots + \binom{n}{n}uv_n$$

$y = e^{m\tan^{-1}(x)}$

$$y_1 = \frac{me^{m\tan^{-1}(x)}}{1+x^2}$$

Then

$y_1(1+x^2) = me^{m\tan^{-1}(x)}$

$y_1(1+x^2) = my$

Differentiate both sides of the equation n times with respect to x

$$\frac{d^n}{dx^n}\left(y_1(1+x^2)\right) = my_n$$

Let $u = y_1, v = 1 + x^2$. Then

$u_n = y_{n+1}$

$u_{n-1} = y_n$

$u_{n-1} = y_{n-1}$

…

$v_1 = (1+x^2)' = 2x$

$v_2 = (2x)' = 2$

$v_3 = (2)' = 0$

…

$$\frac{d^n}{dx^n}\left(y_1(1+x^2)\right) = u_n v + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + 0 =$$

$$= y_{n+1}(1+x^2) + ny_n(2x) + \frac{n(n-1)}{2}y_{n-1}(2)$$

Hence

$$y_{n+1}(1+x^2) + ny_n(2x) + \frac{n(n-1)}{2}y_{n-1}(2) = my_n$$

$$\boldsymbol{y_{n+1}(1+x^2) + y_n(2nx-m) + n(n-1)y_{n-1} = 0}$$