Answer on Question #79033 – Math – Calculus

Question

Show that the functions $f(x,y) = \ln x - \ln y$ and $g(x,y) = \frac{x^2+2y^2}{2xy}$ are functionally dependent.

Solution

Definition

Two functions $f(x,y)$ and $g(x,y)$ defined in a domain D are functionally dependent if there exists a continuously differentiable function $\varphi(u,v)$ such that

1. $\varphi(f(x,y), g(x,y)) \equiv 0$.
2. $\nabla\varphi \neq 0$ at each point $(u,v) = \big(f(x,y), g(x,y)\big)$ for (x,y) in D.

Theorem

Let F be a differentiable transformation defined by

$$F\colon \begin{cases} u = f(x,y) \\ v = g(x,y) \end{cases}$$

If $f(x,y)$ and $g(x,y)$ are functionally dependent, then the Jacobian of F is identically zero.

In this problem,

$$f(x,y) = \ln x - \ln y, \qquad g(x,y) = \frac{x^2+2y^2}{2xy}$$

$x > 0, y > 0$

We start by finding the Jacobian

$$f_x = \frac{1}{x}, f_y = -\frac{1}{y}$$

$$g_x = \frac{\partial}{\partial x}\left(\frac{1}{2y}x + y\left(\frac{1}{x}\right)\right) = \frac{1}{2y} - \frac{y}{x^2}, \qquad g_y = \frac{\partial}{\partial y}\left(\frac{1}{2y}x + y\left(\frac{1}{x}\right)\right) = -\frac{x}{2y^2} + \frac{1}{x}$$

$$\det J = \det\left[\frac{\partial(f,g)}{\partial(x,y)}\right] = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = f_x g_y - f_y g_x =$$

$$= \frac{1}{x}\left(-\frac{x}{2y^2} + \frac{1}{x}\right) - \left(-\frac{1}{y}\right)\left(\frac{1}{2y} - \frac{y}{x^2}\right) =$$

$$= -\frac{1}{2y^2} + \frac{1}{x^2} + \frac{1}{2y^2} - \frac{1}{x^2} \equiv 0$$

Thus we are led to seek a functional relationship. By elementary manipulations we can rewrite f and g in the form,

$$f(x,y) = \ln x - \ln y = \ln\left(\frac{x}{y}\right) \Rightarrow \frac{x}{y} = e^{f(x,y)}$$

$$g(x,y)=\frac{x^2+2y^2}{2xy}=\frac{1}{2}\left(\frac{x}{y}\right)+\frac{1}{\left(\frac{x}{y}\right)}$$

Then

$$g(x,y)=\frac{1}{2}e^{f(x,y)}+e^{-f(x,y)}$$

So that f and g are functionally dependent.