Question

Find the domain and range of the function f defined by $f(x,y) = \frac{3x^2y^2}{x^2+y^4}$.

Solution

$$x^2 + y^4 \neq 0 ,$$

$$\begin{cases} x \neq 0 \\ y \neq 0 \end{cases}$$

Domain: $D(f) = \{f(x,y)|(x,y) \in R/\{(0,0)\}$

Range: $R(f) = \{f(x,y) \geq 0\}$.

Let $\frac{3x^2y^2}{x^2+y^4} = t$.

If $ty^4 - 3x^2y^2 + tx^2 = 0$, then

$D = 9x^4 - 4t^2x^2 = x^2(9x^2 - 4t^2) \geq 0$, hence $4t^2 \leq 9x^2$.

If $ty^4 - 3x^2y^2 + tx^2 = 0$, then $x^2 = \frac{-ty^4}{t-3x^2} \geq 0$, hence $t(t - 3x^2) \leq 0$, therefore, $0 \leq t \leq 3x^2$.