Answer on Question #79031 – Math – Calculus

Question

If $f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$ then show that the repeated limits of f do not exist.

Examine the function f for simultaneous limit at the origin.

Solution

Consider

$$\lim_{y\to0}\left(\lim_{x\to0} f(x,y)\right) = \lim_{y\to0}\left(\lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}\left(x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right)\right)\right)$$

Suppose $y \neq \frac{1}{\pi n}, n \in Z$, then

$$\lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}\left(x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right)\right) = \lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}\left(x\sin\left(\frac{1}{y}\right)\right) + \lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}\left(y\sin\left(\frac{1}{x}\right)\right) =$$

$$= \sin\left(\frac{1}{y}\right)\lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}(x) + y\lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}\left(\sin\left(\frac{1}{x}\right)\right) = 0 + y\lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}\left(\sin\left(\frac{1}{x}\right)\right)$$

We know that $\lim_{x\to0}\left(\sin\left(\frac{1}{x}\right)\right)$ does not exist:

$$x = \frac{1}{\pi k}, k \in Z => \lim_{x\to0}\left(\sin\left(\frac{1}{x}\right)\right) = 0$$

$$x = \frac{1}{\frac{\pi}{2} + 2\pi k}, k \in Z => \lim_{x\to0}\left(\sin\left(\frac{1}{x}\right)\right) = 1$$

Hence $\lim_{\substack{x\to0 \\ y-fixed \\ y\neq0}}\left(x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right)\right)$ does not exist.

Therefore, $\lim_{y\to0}\left(\lim_{x\to0} f(x,y)\right)$ does not exist.

Similarly, $\lim_{x\to0}\left(\lim_{y\to0} f(x,y)\right)$ does not exist.

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ if:

$\forall\varepsilon > 0\ \exists\delta$ such that if $(x,y) \in D(f)$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then

$$|f(x,y) - L| < \varepsilon$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Given $\varepsilon > 0$. Let $\delta = \varepsilon/2$. Then for $0 < |x| < \delta, 0 < |y| < \delta$

$$|f(x,y) - L| = \left|x \sin\left(\frac{1}{y}\right) + y \sin\left(\frac{1}{x}\right) - 0\right| \le |x| \left|\sin\left(\frac{1}{y}\right)\right| + |y| \left|\sin\left(\frac{1}{x}\right)\right| \le$$

$$\le |x| \cdot 1 + |y| \cdot 1 < \delta + \delta = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This means that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$