Answer on Question #78777 - Math - Abstract Algebra
July 2, 2018

Question. If $G = \langle x \rangle$ is of order 25, then x^a generates G, where a is a factor of 25. State whether the given statement is true or false, give reasons for your answer.

Solution. If we interpret the statement as "for every integer a that is a factor of 25, x^a generates G," then the statement is false.

Let $a = 5$ which is a factor of 25. We will prove that some element of G is not in $\langle x^a \rangle$. Every element of $\langle x^a \rangle$ is of the form $(x^a)^n$ for some integer n. We can divide n by 5, so there are integers q and r such that $n = 5q + r$ and $0 \leq r < 5$. We have

$$(x^a)^n = x^{an} = x^{5(5q+r)} = (x^q)^{25} x^{5r} = x^{5r};$$

$(x^q)^{25} = e$ because the order of every element of G divides 25. Hence $\langle x^a \rangle = \{x^0, x^5, x^{10}, x^{15}, x^{20}\}$, and this set contains at most 5 elements. However, G contains 25 elements. Hence some element of G is not in $\langle x^a \rangle$.

Answer. The statement is false.