Answer on Question #78776 – Math – Abstract Algebra

Question

If G is a group such that $ord(G) = 2m$, where $m \in \mathbb{N}$, then G has a subgroup of order m.

State the given statement is true or false, give reasons for your answer.

Solution

1) Assume that $m = 1$. Then there are two subgroups of order 1 and 2.

2) If $m = 2$ (or 3, or 5), then by Cauchy's theorem, G has an element of order 2 (or 3, or 5).

3) If $m = 4$, then by the first Sylow theorem, G has a subgroup of order 4.

4) Let's consider A_4, the group of all even permutations of a 4-element set, i.e. products of an even number of transpositions. The order of G is $\frac{4!}{2} = 12 = 2 * 6$; it consists of the identity, the one fixed points, and the double transpositions. Any subgroup of order 6 contains the identity. By Cauchy's theorem, it also contains an element of order 2 and an element of order 3, i. e. a double transposition and a fixed point, t and f. Therefore such subgroup must contain 2 other fixed points, tf and ft (see the Cayley table: $tf = ft \Longrightarrow (tf)^2 = (ft)^{-1} \Longrightarrow ftf = f^{-1} \Longrightarrow tf = e \Longrightarrow |tf| \neq 3$). By the inverse element axiom, it must contain 2 other fixed points, $(ft)^{-1}$ and $(tf)^{-1}$ (this can also be directly proven). This is a contradiction $(1 + 2 + 2 + 2 > 6)$.

	()	(123)	(124)	(132)	(134)	(142)	(143)	(234)	(243)	(12)(34)	(13)(24)	(14)(23)
()	()	(123)	(124)	(132)	(134)	(142)	(143)	(234)	(243)	(12)(34)	(13)(24)	(14)(23)
(123)	(123)	(132)	(13)(24)	()	(234)	(143)	(14)(23)	(12)(34)	(124)	(134)	(243)	(142)
(124)	(124)	(14)(23)	(142)	(134)	(13)(24)	()	(243)	(123)	(12)(34)	(143)	(132)	(234)
(132)	(132)	()	(243)	(123)	(12)(34)	(14)(23)	(142)	(134)	(13)(24)	(234)	(124)	(143)
(134)	(134)	(124)	(12)(34)	(14)(23)	(143)	(234)	()	(13)(24)	(132)	(123)	(142)	(243)
(142)	(142)	(234)	()	(13)(24)	(132)	(124)	(12)(34)	(14)(23)	(143)	(243)	(134)	(123)
(143)	(143)	(12)(34)	(123)	(243)	()	(13)(24)	(134)	(142)	(14)(23)	(124)	(234)	(132)
(234)	(234)	(13)(24)	(134)	(142)	(14)(23)	(12)(34)	(123)	(243)	()	(132)	(143)	(124)
(243)	(243)	(143)	(14)(23)	(12)(34)	(124)	(132)	(13)(24)	()	(234)	(142)	(123)	(134)
(12)(34)	(12)(34)	(243)	(234)	(143)	(142)	(134)	(132)	(124)	(123)	()	(14)(23)	(13)(24)
(13)(24)	(13)(24)	(142)	(143)	(234)	(243)	(123)	(124)	(132)	(134)	(14)(23)	()	(12)(34)
(14)(23)	(14)(23)	(134)	(132)	(124)	(123)	(243)	(234)	(143)	(142)	(13)(24)	(12)(34)	()

Answer: The statement is false. The smallest counter example is A_4 $(m = 6)$.