Answer on Question #78753 - Math - Abstract Algebra
June 30, 2018

Question. Let G be a group, $H \trianglelefteq G$ and $\beta \leq (G/H)$. Let $A = \{x \in G \mid Hx \in \beta\}$. Show that

1. $A \leq G$,
2. $H \trianglelefteq A$,
3. $\beta = (A/H)$.

Answer.

1. We will show that A is closed under group operations of G.
   - Let $x, y \in A$. Then $Hx, Hy \in \beta$ by the definition of A. As $\beta \leq (G/H)$, $HxHy \in \beta$. As $H \trianglelefteq G$, $HxHy = HHxy = Hxy$. Hence $xy \in A$ by the definition of A.
   - As $He \in \beta$, $e \in A$.
   - Let $x \in A$. Then $Hx \in \beta$ by the definition of A. As $\beta \leq (G/H)$, $x^{-1}H = x^{-1}H^{-1} = (Hx)^{-1} \in \beta$. As $H \trianglelefteq G$, $x^{-1}H = Hx^{-1}$. Hence $x^{-1} \in A$ by the definition of A.
2. Let $x \in A$. Then $x \in G$, and $xH = Hx$ because H is normal in G. Therefore, H is normal in A.
3. We know that $(A/H) = \{Hx \mid x \in A\}$. We will prove $\beta = (A/H)$ which is equivalent to

   $$x' \in \beta \iff x' \in (A/H)$$

   for every $x' \in (G/H)$.
   - ($\implies$) Let $x' \in \beta$. Then $x' = Hx$ for some $x \in G$ because $\beta \leq (G/H)$. Then $x \in A$ by the definition of A. Therefore, $x' \in (A/H)$ by the definition of (A/H).
   - ($\impliedby$) Let $x' \in (A/H)$. Then $x' = Hx$ for some $x \in A$ by the definition of (A/H). Therefore, $x' \in \beta$ by the definition of A.