Answer on Question #78752 - Math - Abstract Algebra
June 30, 2018

Question. For x belonging to G, define $H_x = \{g^{-1}xg \mid g \in G\}$. Under what conditions on x will $H_x \leq G$? Further, if $H_x \leq G$, will $H_x \trianglelefteq G$? Give reason for your answer.

Solution. Assume that H_x is a subgroup of G (denoted by $H_x \leq G$). Then $e \in H_x$. So there is $g \in G$ such that $e = g^{-1}xg$, but then $e = geg^{-1} = gg^{-1}xgg^{-1} = x$.

On the other hand, assume that $x = e$.

- We have $e = e^{-1}xe \in H_x$.
- For every $y \in H_x$, $y = g^{-1}xg$ for some $g \in G$, so $y = g^{-1}eg = e$.

Hence $H_x = \{e\}$. This is a trivial subgroup of G, so $H_x \leq G$.

We see that $H_x \leq G$ if and only if $x = e$.

The subgroup $\{e\}$ is also normal in G because for every $g \in G$, $g^{-1}eg = e \in \{e\}$.

Answer. $H_x \leq G$ if and only if $x = e$, and in that case, $H_x \trianglelefteq G$.